We consider the product of the polynomials

\[ F = f_0 + f_1 X + \cdots + f_{n-1} X^{n-1}, \]
\[ G = g_0 + g_1 X + \cdots + g_{n-1} X^{n-1}, \]

where \( n \geq 1 \). When the polynomials have different degrees, we should add zeros to the polynomial of lowest degree. The degree is \( n - 1 \), but the number of coefficients for each polynomial is \( n \).

Notations:

\[ \deg(F) = \deg(G) = n - 1, \quad \text{len}(F) = \text{len}(G) = n, \]

will be used hereafter.

The computation of the product \( FG \), which has the properties

\[ \deg(FG) = 2(n - 1), \quad \text{len}(FG) = 2n - 1, \]

(I feel lost when there is no text after equations!)
Let’s now cut $F$ and $G$ in two at $k = \lceil n/2 \rceil$

\[ F = F_0 + F_1 X^k, \quad G = G_0 + G_1 X^k, \]

and we have

\[ \deg(F_0) = \deg(G_0) = k - 1, \quad \deg(F_1) = \deg(G_1) = n - k - 1, \]
\[ \text{len}(F_0) = \text{len}(G_0) = k, \quad \text{len}(F_1) = \text{len}(G_1) = n - k. \]

This decomposition is unique.

By adding zeros again, we can assume that $n$ is even. Then we have

\[ \text{len}(F_1) = \text{len}(F_0), \quad \text{len}(G_1) = \text{len}(G_0). \]

(How about you?)
By this decomposition, the product writes

\[ FG = F_0 G_0 + (F_0 G_1 + F_1 G_0) X^k + F_1 G_1 X^{2k} \]
\[ = H_0 + H_1 X^k + H_2 X^{2k}, \]  \hfill (1)

and the middle term can be written

\[ H_1 = (F_0 G_1 + F_1 G_0) = (F_0 + F_1)(G_0 + G_1) - F_0 G_0 - F_1 G_1, \]

which trades operations \((1 +, 2 \times)\) for \((4 +, 1 \times)\). The sizes are

\[ \text{deg}(H_0) = \text{deg}(H_1) = \text{deg}(H_2) = 2(k-1), \]
\[ \text{len}(H_0) = \text{len}(H_1) = \text{len}(H_2) = 2k - 1. \]

Remark: in Eq. (1), the high-order terms of \(H_0\) spread over the low-order terms of \(H_1\), and the high-order terms of \(H_1\) spread over the low-order terms of \(H_2\).
To define our recursive algorithm, we also need that sizes of the polynomials $F_0$, $F_1$, $G_0$, and $G_1$, i.e. $k$ to be even. This leads to $n$ being a power of two.

To design our algorithm, we consider three arrays named $p$, $q$, and $r$. Their sizes are $4k$, $4k$, and $8k$, respectively. The diagram below illustrates the initial memory state.

Each box represents a subarray of size $k$. The initial polynomials have been placed at the right half of $p$ and $q$, and hatched boxes represent free boxes. The product will be placed on the left half of $r$. 
In the special case where $k = 1$, we compute the result directly:

\[
\begin{align*}
    r[0] &= p[2] \times q[2]; \\
\end{align*}
\]

Otherwise, follows these steps.
1. Apply algorithm to the following sub arrays

\[
\begin{array}{c}
p: \\
\text{F}_0 \quad \text{F}_1 \\
q: \\
\text{G}_0 \quad \text{G}_1 \\
r: \\
\text{F}_0 \text{G}_0 \\
\end{array}
\]

which places \( \text{F}_0 \text{G}_0 \) in the \( r \) array. Note that we need some free space to perform the sub computation (hatched in red).

2. Copy \( \text{F}_0 \) and \( \text{G}_0 \)

\[
\begin{array}{c}
p: \\
\text{F}_0 \quad \text{F}_0 \quad \text{F}_1 \\
q: \\
\text{G}_0 \quad \text{G}_0 \quad \text{G}_1 \\
\end{array}
\]
3. Apply algorithm to the following sub arrays

- **p:**
  
  | \_ | \_ | \_ |
  |____|____|____|
  | \_ | \_ | \_ |
  |____|____|____|
  | \_ | \_ | \_ |
  |____|____|____|

- **q:**
  
  | \_ | \_ | \_ |
  |____|____|____|
  | \_ | \_ | \_ |
  |____|____|____|
  | \_ | \_ | \_ |
  |____|____|____|

- **r:**
  
  | \_ | \_ | \_ |
  |____|____|____|
  | \_ | \_ | \_ |
  |____|____|____|
  | \_ | \_ | \_ |
  |____|____|____|

which places \( F_1G_1 \) in the \( r \) array and erase original \( F_0 \) and \( G_0 \) data.

4. Add \( F_1 \) to \( F_0 \) and \( G_1 \) to \( G_0 \)

- **p:**
  
  | \_ | \_ | \_ |
  |____|____|____|
  | \_ | \_ | \_ |
  |____|____|____|
  | \_ | \_ | \_ |
  |____|____|____|

- **q:**
  
  | \_ | \_ | \_ |
  |____|____|____|
  | \_ | \_ | \_ |
  |____|____|____|
  | \_ | \_ | \_ |
  |____|____|____|

\( \text{add} \)
5. Apply algorithm to the following sub arrays

\[ p: \begin{array}{c|c}
F_0 + F_1 & F_1 \\
\end{array} \]

\[ q: \begin{array}{c|c}
G_0 + G_1 & G_1 \\
\end{array} \]

\[ r: \begin{array}{c|c|c}
F_0G_0 & F_1G_1 & (F_0 + F_1)(G_0 + G_1) \\
\end{array} \]

which places \((F_0 + F_1)(G_0 + G_1)\) in the \(r\) array.

6. Subtract \(F_0G_0\) and \(F_1G_1\) to the new term to obtain \(H_1\)
7. Finally add $H_1$ at the middle of $(H_0, H_2)$.

The result is hold in the four boxes in the left half of $r$. 