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Lecture 1. Chebyshev points, interpolants, polynomials, and series

Read: *ATAP* chaps 1-3

1. Chebyshev points

$x_j = \cos(j\pi/n),\ j = 0, \dots, n$

```
xn = chebpts(n)
```

Setting: $[-1,1]$. Other intervals $[a,b]$ are handled by the obvious change of variables.

2. Chebyshev interpolants

$p_n =$ the unique polynomial of degree n through data $f_0, \dots, f_n$ at Chebyshev points.

```
pn = chebfun(fn)
```

3. Chebyshev polynomials

$T_k(x) = \cos(k \operatorname{acos}(x))$ This is $\cos(k\theta)$ transplanted by $x = \cos(\theta)$.

```
Tk = chebpoly(k)
```

$T_0(x) = 1, T_1(x) = x, T_2(x) = 2x^2 - 1, T_3(x) = 4x^3 - 3x, \dots$

$T_{k+1} = 2xT_k(x) - T_{k-1}(x) \quad (k \geq 1)$

4. Chebyshev series and coefficients

$f(x) = \sum_{k=0}^{\infty} a_k T_k(x),\ \ a_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1-x^2}}\,dx$ (half this for a_0)

Absolutely and uniformly convergent if f is Lipschitz continuous.

5. Chebfun

$f_C(x) = \sum_{k=0}^{n} c_k T_k(x)$, a Chebyshev interpolant with n chosen adaptively large enough so that $f(x) - f_C(x) = O(10^{-16})$.

```
f = chebfun('f(x)')
n = length(f)-1
c = chebcoeffs(f)
plotcoeffs(f)
```

6. Context

Euler/Gauss and today's mathematicians	E & G were expert in theory, calculation, and applications. Few are today.
Pure and applied mathematics	As scientific fields grow, bifurcations happen.
Discrete and continuous	An equally big distinction.
Theoretical and computational mathematics	Numerical analysis = computational continuous mathematics.
Matlab and Chebfun	Matlab: discrete vectors and matrices. Chebfun: continuous functions and operators.
Real and floating-point numbers	Each $+, -, \times, \div$ entails a relative error of $O(10^{-16})$.
Floating-point arithmetic and Chebfun	Floating-point arithmetic: rounding of numbers. Chebfun: rounding of functions.

7. Functions, series, interpolants, and analysis

Periodic and nonperiodic functions	A priori, there is no reason to expect a function to be periodic.
Trigonometric and algebraic polynomials	Linear combinations of $\sin(k\theta)$ and $\cos(k\theta)$ vs. x^k.
Monomial and Chebyshev bases	$\{x^k\}$ is mathematically simple, but hopeless for computation. Need $\{T_k(x)\}$ instead.
Fourier and Chebyshev series	Fourier: trig polys, periodic. Chebyshev: Cheb polys, nonperiodic. Equivalent via $x = \cos(\theta)$.
Equispaced and clustered interpolation points	Equispaced points are hopeless for polynomial interpolation. Need clustered points instead.
Fourier and Chebyshev analysis	"Fourier analysis" exists, "Chebyshev analysis" does not. The reasons are (partly) above.

8. Computing with numbers and with functions

Matlab: vectors, matrices, floating point arithmetic   Chebfun: functions, roots, integrals, derivatives, extrema