$\boxed{1.}$ Suppose $p(x)=\sum_{k=0}^{n} a_k x^k$, and $a_n=1$. There is the companion matrix C.

eigenmachin   validity

$$\underbrace{\begin{bmatrix}0&1&&\\&0&1&\\&&\ddots&1\\-a_0&\cdots&-a_{n-2}&-a_{n-1}\end{bmatrix}}_{C}\begin{bmatrix}1\\x\\\vdots\\x^{n-1}\end{bmatrix}= x\begin{bmatrix}1\\x\\\vdots\\x^{n-1}\end{bmatrix}\begin{matrix}\checkmark\\\checkmark\\\vdots\\\text{ok if } p(x)=0\end{matrix}$$

This equality is true only if $p(x)=0$ in the last row. So $\{\text{roots of } p\}=\{\text{eigenvalues of } C\}$

There is a Tchebyshev analog, the colleague matrix.

$$\begin{bmatrix}0&1&&&\\\frac{1}{2}&0&\frac{1}{2}&&\\&&\ddots&&\\-\frac{a_0}{2}&-\frac{a_1}{2}&\cdots&&-\frac{a_{n-1}}{2}\end{bmatrix}\begin{bmatrix}T_0(x)\\T_1(x)\\\vdots\\T_{n-1}(x)\end{bmatrix}=x\begin{bmatrix}T_0(x)\\T_1(x)\\\vdots\\T_{n-1}(x)\end{bmatrix}\begin{matrix}\checkmark\\\checkmark\\\vdots\\p(x)=0\end{matrix}$$

Computing eigs in Matlab is $O(n^2)\sim O(n^3)$ (QR algorithm, 1960)*

Chebfun roots

If $n\leq$ few hundreds (200), then the above algorithm is used. Else, the domain is subdivised.

* Transform the matrix to its Hessenberg form, then the QR iterations makes eigenvalues appear in the diagonal.

$\boxed{3.1}$ Classical quadrature: $I=\int_{-1}^{+1} f(x)\,dx$ gives a quad. formula $I_n=\sum_{k=0}^{n} \overset{\text{weights}}{w_k} f(\underset{\text{nodes}}{x_k})$

Choice of weights: • integrate the polynomial interpolant of f through $f(x_k)$

• it is exact if f is a polynomial of degree $\leq n$.

• if $\{x_k\}$ is equally spaced: Newton-Cotes quad. (17th); if $\{x_k\}$ Chebyshev: Clenshaw-Curtis quad. (1960); if $\{x_k\}$ roots of Legendre polynomial: Gauss quad. (1814).

NC quad. bad for large n, even for analytic functions

CC and G quad. always converge if f is continuous. They are not very different in pract.

• With G, quad. is exact if f is a poly of degree $\leq 2n+1$ "This statement is misleading"

$\boxed{\text{Thm 19.3}}$ Suppose f is analytic and bounded in E_ρ:

$$CC:\ |I-I_n|=O(\rho^{-n}) \qquad G:\ |I-I_n|=O(\rho^{-2n})$$

Outline of Proof: CC is exact for poly of deg $\leq n$ such has the truncated Cheby series

$$|I-I_n|=|I(f)-I_n(f)|=|I(f-f_n)-I_n(f-f_n)|\leq \underbrace{|I(f-f_n)|}_{O(\rho^{-n})}+\underbrace{|I_n(f-f_n)|}_{O(\rho^{-n}) \text{ since } \sum w_k=\sum|w_k|=2}$$

$(-I(f_n)+I_n(f_n)=0)$

The 2nd term fails to be $O(\rho^{-n})$ for NC quad.

Computing nodes and weights (Legendre) roots of P_{n+1} are roots of $0+0P_1+0P_2+\dots+P_{n+1} \Rightarrow$ Colleague matrix

$$\begin{bmatrix}0&\times&&\\\times&0&\times&\\&&\ddots&\times\\&&\times&0\end{bmatrix}$$ Jacobi matrix, symmetric | eigenvalues $\rightarrow$ roots, eigenvectors $\rightarrow$ weights

$O(n^2)$ works (Matlab $O(n^3)$) and $O(n)$ alternatives!