VALIDATED NUMERICS #2 by W.T.

Hausdorff distance

$$d(\mathbb{I}a, \mathbb{I}b) = \max\{|\underline{a}-\underline{b}|, |\overline{a}-\overline{b}|\}$$

Property: (convergence) $\lim_{k\to+\infty} \mathbb{I}a_k = \mathbb{I}a \iff \lim_{k\to+\infty} d(\mathbb{I}a_k, \mathbb{I}a) = 0$

Set operations: $\mathbb{I}a \cap \mathbb{I}b$ may be empty.

Set arithmetic operations: "It's better to check the signs than comparing all four possibilities"

$\mathbb{IR}$-arithmetics inherits; commutative law for + and ×, associative law for + and ×.
But the distributive law does not always hold.

$$\mathbb{I}a \times (\mathbb{I}b + \mathbb{I}c) \subseteq \mathbb{I}a \times \mathbb{I}b + \mathbb{I}a \times \mathbb{I}c,$$

and there is no additive/multiplicative inverse.

Division extension: $\mathbb{I}a \div \mathbb{I}b = \{c \in \mathbb{R} : bc = a,\ a \in \mathbb{I}a \text{ and } b \in \mathbb{I}b\}$

Exemple: $[1,2] \div [-5,3] = [1,2] \div [-5,0) \cup [1,2] \div \{0\} \cup [1,2] \div (0,3]$

$$= (-\infty, -\tfrac{1}{5}] \cup \emptyset \cup [\tfrac{1}{3}, +\infty) = (-\infty, -\tfrac{1}{5}] \cup [\tfrac{1}{3}, +\infty) = \mathbb{R} \setminus (-\tfrac{1}{5}, \tfrac{1}{3})$$

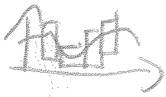
This tells that result cannot be close to zero. But result is not an interval.

"You really have to ~~write~~ write your $\mathbb{IR}$ code from scratch. Converting an old $\mathbb{R}$ code usually dosen't give satisfying results"

- If $\mathbb{I}x = [-5, 3]$, what do we mean by $\mathrm{sqrt}(\mathbb{I}x)$? Error? Complex result?

An elementary function is a finitely many composition of std functions with $\{+, -, \times, \div, \circ\}$.

The inclusion principle $\quad f(\mathbf{x}) \supseteq R(f, \mathbf{x})$

Key-idea: Adaptive subdivision $\Rightarrow$ Controlled discretization 

Interval continuity $\quad \mathrm{rad}(\mathbf{x}) \to 0 \Rightarrow \mathrm{rad}(f(\mathbf{x})) \to 0$

"In practice, it does not hold" "You have to set a good stopping condition"

Inclusion monotonicity $\quad \mathbf{x}_1 \subseteq \mathbf{x}_2 \Rightarrow f(\mathbf{x}_1) \subseteq f(\mathbf{x}_2)$

"A very desirable property, but not always needed"

Interval Lipschitz $\quad \mathrm{rad}(f(\mathbf{x})) \leq K \times \mathrm{rad}(\mathbf{x})$, w/ K indp. of $\mathbf{x}$.

"A nice property"

Application:
$$R(f; \mathbf{x}) = R(f, \bigcup_1^N \mathbf{x}_i) = \bigcup_1^N R(f, \mathbf{x}_i)$$
$$\subseteq \bigcup_1^N f(\mathbf{x}_i) \subseteq f(\bigcup_1^N \mathbf{x}_i) = f(\mathbf{x})$$

Inclusion principle $\quad$ Inclusion monotonicity

"~~Exclusion principle~~" $\quad f(\mathbf{x}) \supseteq R(f, \mathbf{x})$

"Exclusion principle" $\quad y \notin f(\mathbf{x}) \Rightarrow y \notin R(f, \mathbf{x})$. (Take $y=0$)