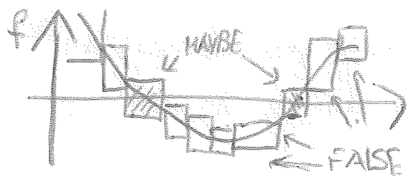
PRACTICAL PROBLEM SOLVING by W.T.

1. Nonlinear equations Solve $f(x)=0$.

* Interval bisection — indirect (no guaranties), robust and global

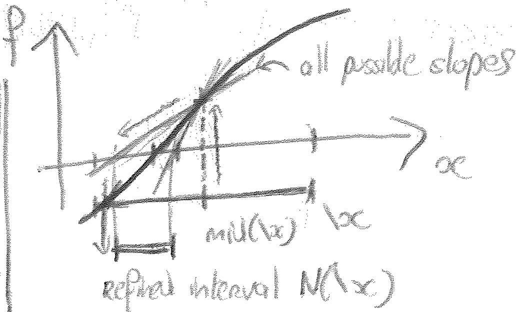
Based on (3-1)-fold logic (TRUE/FALSE/MAYBE) but uses only the two last folds ($0 \notin f(\mathbf{x}_i)$ / $0 \in f(\mathbf{x}_i)$)



Improvements

- When $f(\underline{x}_i) f(\overline{x}_i) < 0$, you know that you have a zero in it. If in addition $f'(\mathbf{x}_i) \neq 0$, this zero is unique. This way you can count roots.

* Interval Newton method Let $f \in C^1$ and assume f has a simple zero $x^* \in \mathbf{x}$. By the Mean Value theorem $f(x) = f(x^*) + f'(\xi)(x - x^*)$ with $\xi \in \mathbf{x}$. Now assume that $f'(x) \neq 0 \ \forall x \in \mathbf{x}$, then $x^* = x - f(x)/f'(\xi) \in x - f(x)/f'(\mathbf{x}) = N(\mathbf{x}, x, f)$. Often, $x = \mathrm{mid}(\mathbf{x})$ and we simply write $N(\mathbf{x})$.



Theorem Let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Given a 'box' $\mathbf{x} \in \mathbb{IR}^n$ and $x_0 \in \mathbf{x}$. Let $f'(\mathbf{x}) \supseteq R(f', \mathbf{x})$. Assume that $[f'(\mathbf{x})]^{-1}$ exists, and set

$$N(\mathbf{x}) = N_f(\mathbf{x}, x_0) = x_0 - [f'(\mathbf{x})]^{-1} f(x_0)$$

1. If $N(\mathbf{x}) \cap \mathbf{x} = \emptyset$, then $f|_{\mathbf{x}} \neq 0$
2. If $N(\mathbf{x}) \subseteq \mathbf{x}$, then f has a unique zero in $\mathbf{x}$.

$\neg 1$ & $\neg 2$ $\Rightarrow$ Continue with $\mathbf{x} \cap N(\mathbf{x})$.

"If $[f'(\mathbf{x})]^{-1}$ exists and 2., then you're in a 'safe' region"

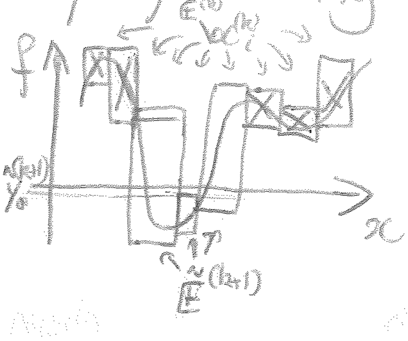
"You have the choice to stop as soon as you know that the zero is unique, OR to continue shrink the interval until the desired accuracy"

2. Optimization (unconstrained) Given $f: \mathbf{x} \to \mathbb{R}$, find:

$$y^* = y^*(f, \mathbf{x}) = \min\{f(x) : x \in \mathbf{x}\}, \quad \mathbb{E}^* = \mathbb{E}^*(f, \mathbf{x}) = \{x^* \in \mathbf{x} : f(x^*) = y^*\}$$

Assumption: f is C^0 ($\mathbf{x}$ is compact) $\Rightarrow$ y^* and $\mathbb{E}^*$ exist.

Strategy Find decreasing upper bound $\tilde{y}^{(k)}$ for y^* by removing $\tilde{E}^{(k)}$ subsets of $[x]$.



Start $(k=0)$ — $\tilde{E}^{(0)} = [x]$ and $\tilde{y}^{(0)} = +\infty$

Stage k — $\tilde{E}^{(k)} = \{[x_1^{(k)}], \dots, [x_{N_k}^{(k)}]\}$ and $\tilde{y}^{(k)} \in \mathbb{R} \cup \{+\infty\}$

Stage $k+1$ — $\tilde{y}^{(k+1)} = \min\{\tilde{y}^{(k)}, \overline{[y_i]}\}$ where $[y_i] = f([x_i^{(k)}])$

$\tilde{E}^{(k+1)}$ containes subdivised $[x_i^{(k)}]$ where $\underline{[y_i]} \leq \tilde{y}^{(k+1)}$

At stage $k+1$ you keep only boxes $[x_i^{(k)}]$ whose lower bound is less or equal than the ~~minimum~~ lowest upper bound of all the boxes. And you split your new boxes.

It may occur that no boxes are discarded. In this case, evaluate, select the $[x_i^{(k)}]$ where $\mathrm{mid}([x_i^{(k)}])$ is the lowest and use the midpoint value as $\tilde{y}^{(k+1)}$.

If f is monotone over $[x_i^{(k)}]$ ($f'|_{[x_i^{(k)}]} \neq 0$), you can discard this box.

If f is concave over $[x_i^{(k)}]$ ($f''|_{[x_i^{(k)}]} < 0$), you can discard this box.

The last stage gives you an enclosure of minimum position(s) and value. You will not lost minimum positions if there is more than one (e.g. cos).

3. Quadrature Goal $I = \int_a^b f(x)dx$. You start by the classical trick of subdivising ~~your f~~ $[a,b]$ into subintervalls, integrate many times, and sum the results.

$$I = \sum_{i=1}^{N} \int_{[x_i]} f(x)dx \in \sum_{i=1}^{N} \int_{[x_i]} f([x_i])dx = \sum_{i=1}^{N} \mathrm{width}([x_i]) f([x_i])$$

You use lower/upper bound of the boxes just as in the Riemann integral (without going to the lim).

1st order method: increase N. (Converges slowly). Useful if only $\mathrm{sign}(I)$ is needed.

Recall (Taylor) $f(x) \in \sum_{k=0}^{n-1} f_k (x-\tilde{x})^k + \frac{1}{n!} f^{(n)}([x])(x-\tilde{x})^n$ $\quad x, \tilde{x} \in [x]$, f_k obtained by TA.

If we note $\varepsilon_n = \mathrm{mag}\left(\frac{1}{n!} f^{(n)}([x]) - \frac{1}{n!} f^{(n)}(\tilde{x})\right)$, then $\boxed{f(x) \in \sum_{k=0}^{n-1} f_k (x-\tilde{x})^k + [-\varepsilon_n, +\varepsilon_n](x-\tilde{x})^n}$

easy to integrate — small enclosure

$$I = \int_{\tilde{x}-R}^{\tilde{x}+R} f(x)dx \in \sum_{k=1}^{N} f_k \int_{\tilde{x}-R}^{\tilde{x}+R} (x-\tilde{x})^k dx + [\varepsilon_n] \int_{\tilde{x}-R}^{\tilde{x}+R} |x-\tilde{x}|^n dx$$

High order approx. and interval bound.