DYNAMICAL SYSTEMS I by W.T.

<u>Overall goal</u>: understand long-term behavior of a given system.

Two categories: discrete d.s. and continuous d.s.

Discrete: given x_0 and $\overset{\text{parameters}}{p}$, study $x_{k+1}=f(x_k,p)$, and what is $\lim\limits_{k\to+\infty} x_k$?

Continuous: given x_0 and $\overset{\text{parameters}}{p}$, study $\underset{\varphi_t(x_0,p)}{\cancel{\varphi_{x_0,p}(t)}}$ as $t\to+\infty$ where $\{\dot\varphi=f(\varphi_t),\ \varphi_0=x_0\}$ (Init. val. pb.)

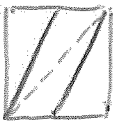
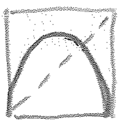
$\boxed{\text{Definition}}$ (discrete) x is a periodic pt. of period p of f if $\overbrace{f\circ f\circ\cdots\circ f}^{p\text{ times}} = f^p(x)=(x)$

"Composing p time magnify nonlinearity"
"Taking p higher and higher is a num. challenge"

Finding a period-p pt of f is the same as solving $f^p(x)=0$ or $g(x)=f^p(x)-x=0$

<u>Examples</u>: $f_p:[0,1]\circlearrowleft$ (into itself), $f_1(x,p)=px(1-x)$, $p\in[0,4]$ "logistic map"

$f_2(x,p)=2x \bmod 1$

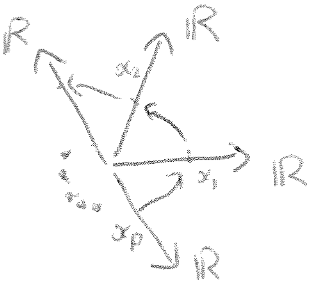
f_2  f_1 

$\boxed{\text{Stability}}$ Note that $(f^p)'(x_0)=\prod_{k=0}^{p-1} f'(x_k)$ is numerically well-posed.

<u>Shooting approach</u> (Swap composition for dimension) Global map: $F:\mathbb{R}^p\circlearrowleft$ such that

$F_1(x)=x_2-f(x_1)$, $F_2(x)=x_3-f(x_2)$, …, $F_p(x)=x_1-f(x_p)$

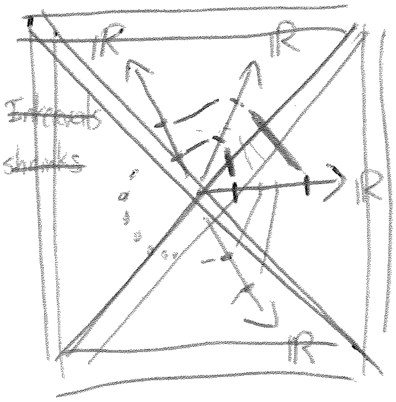
<u>How?</u> Find $\mathbf{x}\in\mathbb{IR}^p$ such that $N_F(\mathbf{x})\subseteq\mathbf{x}$, then $\exists!\, x^*\in\mathbf{x}$ s.t. $F(x^*)=0$



<u>Note</u> $F(x)=0 \iff x_1$ is p-periodic (so is $x_2, x_3, \dots, x_p$)

Now solve $F(x)=0$ via Interval Newton(-Cotes?) method.

<u>Recall</u> $N_F(\mathbf{x})=\overset{\text{midpoint of }\mathbf{x}}{\check{x}}-[DF(\mathbf{x})]^{-1}F(\check{x})$



We use the fact that F is "sparse"

$$DF(x)=\begin{bmatrix} -f'(x_1) & 1 & 0 & \\ 0 & -f'(x_2) & 1 & \\ & & \ddots & \ddots \\ 1 & 0 & & \end{bmatrix}$$

$[DF(x)]^{-1}$ is dense :-(

But, we only need the product $[DF(x)]^{-1}F(\check{x})$, which we note $\mathbf{h}$ and can be written

$F(\check{x})=[DF(x)]\,\mathbf{h}$

$$\begin{cases} \check{x}_2-f(\check{x}_1)=-f'(x_1)h_1+h_2 \\ \check{x}_3-f(\check{x}_2)=-f'(x_2)h_2+h_3 \\ \vdots \\ \check{x}_{\cancel{p}}-f(\check{x}_p)=-f'(x_p)h_p+h_1 \end{cases}$$

Can be explicitely solved for $\mathbf{h}$.

Now ~~we~~ we can put intervals into it.

Although p is free. Take $x_0 \in I$ and compute $x_1, \ldots, x_N$, N huge. Perhaps $|x_0 - x_M| < \delta$, then M may be a period, Now you can apply interval Newton to this candidate and see if it converges.