On Rejection Sampling in Lyubashevsky's Signature Scheme

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Our Questions

1. Can we make rejection sampling faster?
2. How compact can Lyubashevsky's signatures get?
3. How to reach this compactness?
4. Bonus:
    4.1 Are the proofs in the litterature flawed?
    4.2 Similar questions for the BLISS `(Ducas et al.; Crypto'13)` variant
5. Can we do rejection sampling with bounded runtime?

Motivations

- Already implemented in practice.
- NIST PQC standardisation project finalist:
- Rejection sampling has been widely used since its introduction in cryptography (Lyubashevsky; AC'09)...
- ... but mostly in a black-box manner, and only with very few distributions.

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(Lyubashevsky; AC'09), (Lyubashevsky; EC'11)...

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Digital Signature

- KeyGen
  - **Input**: Security parameter 1^λ
  - **Output**: Signing key sk and verification key vk

- Sign
  - **Input**: sk and message μ
  - **Output**: Signature σ

- Verify
  - **Input**: vk and μ and σ
  - **Output**: True or False

Properties of Digital Signatures

- Correctness: Verify(vk , μ , Sign(sk , μ)) returns False with negligible probability.

- Unforgeability: Without sk , it is hard to produce an unseen valid pair (μ^* , σ^*) even with a signing oracle.

Security Assumption

$\mathrm{SIS}_{n,m,\beta}$

Given uniform $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, find nonzero $\mathbf{s} \in \mathbb{Z}_q^m$ s.t. $\|\mathbf{s}\| \leq \beta$ and

$$\mathbf{A}\,\mathbf{s} = 0$$

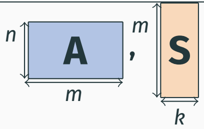
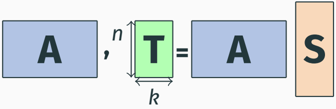
- Post-quantum assumption based on Euclidean Lattices.
- Gets harder when β is smaller.

Lyubashevsky's Signature Scheme

(Lyubashevsky; AC'09),

(Lyubashevsky; EC'11)...

KeyGen

sk	vk
	
where **S** is “small.”	

Sign

On input sk, μ, sample a small

$$\mathbf{y} \hookleftarrow Q.$$

A signature σ for a message μ is of the form

$\mathbf{c}$	$\mathbf{z}$
$\mathbf{c} = \mathbf{H}\left(\mathbf{A}\mathbf{y}, \mu\right)$	$\mathbf{y} + \mathbf{S}\mathbf{c}$

Verify

vk	**c**	**z**
$\mathbf{A}$, $\mathbf{T} = \mathbf{A}\mathbf{S}$	$H(\mathbf{A}\mathbf{y}, \mu)$	$\mathbf{y} + \mathbf{S}\mathbf{c}$

To verify, check that $\|\mathbf{z}\| \leq \gamma$ and that

$$\mathbf{c} = H\left(\mathbf{A}\mathbf{z} - \mathbf{T}\mathbf{c}, \mu\right)$$

Properties

Correctness

$\forall \mathbf{S}, \mathbf{c} : \Pr_{\mathbf{y} \hookleftarrow Q}(\|\mathbf{y} + \mathbf{S}\mathbf{c}\| > \gamma) \leq \mathrm{negl}(\lambda) \implies$ the scheme is correct.

- Instantiated with Q either Gaussian or Hypercube-Uniform,
- This version is not secure:
  the distribution of $\mathbf{z}$ heavily depends on $\mathbf{S}$.

Making the Scheme secure

Ideal signature:

$\mathsf{Sign}_2(\mu, \mathbf{A}, \mathbf{S})$:

1: $\mathbf{z} \hookleftarrow P$
2: $\mathbf{c} \leftarrow U(\mathcal{C})$
3: set $H(\mathbf{Az} - \mathbf{Tc}, \mu) = \mathbf{c}$
4: return $(\mathbf{z}, \mathbf{c})$

Intuition

- If Q Gaussian, take $P = Q$ with standard deviation such that the shift $\mathbf{Sc}$ is not noticeable.
- If $Q = U([-\gamma_1, \gamma_1]^m)$, reject the signature if $\mathbf{z} \notin [\gamma_2, \gamma_2]^m$.
  - Value of γ_1, γ_2?
  - Generalise for Gaussians and other distributions?

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Two solutions:

1. Flooding
   - Set the standard deviation of Q really large
   - Consequence: γ is really large
   - Used by (Damgård et al.; CRYPTO12), (Agrawal et al.; ICALP22)
2. Rejection Sampling (Lyubashevsky; AC'09)

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Results on Rejection Sampling

Rejection Sampling for Lyubashevsky's Signatures

- Widely studied and folklore technique from probabilities
- Turns the distribution of $(\mathbf{z}, \mathbf{c})$ into $P \otimes U(\mathcal{C})$
- First used for signatures in `(Lyubashevsky; AC'09)`

Is the way we use rejection sampling "optimal" (in some sense)?
What distributions can we use?

Context

- Given access to many samples distributed following $\tilde{D}$...
- ... How to find a sample distributed following D?
- ⚠ Without modifying the samples!

Setting

- $D, \tilde{D}$ two probability distributions,
- $X_1, \ldots, X_i, \ldots,$ i.i.d. random variables following $\tilde{D}$.

Rejection Sampling Strategy

A family $(A_i : \mathrm{Supp}(\tilde{D})^i \to [i] \cup \{\perp\})_{i \geq 1}$ of randomized algorithms such that $X_J \hookleftarrow D$, where

- $i^* = \min\{i | A_i(X_1, \ldots, X_i) \neq \perp\}$,
- $J = A_{i^*}(X_1, \ldots, X_{i^*})$.

Goal: minimize $\mathbb{E}(i^*)$.

Standard Rejection Sampling

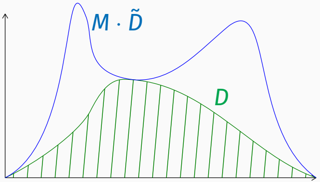

Figure 1: Acceptance zone and sampling domain

Standard Rejection Sampling

$$A_i : (X_1, \ldots, X_i) \mapsto \begin{cases} i & \text{w.p. } \frac{D(X_i)}{M \cdot \tilde{D}(X_i)}, \\ \perp & \text{otherwise.} \end{cases}$$

Works if $D(x) \leq M \cdot \tilde{D}(x)$ for all x. In this case $\mathbb{E}(i^*) = M$.

Imperfect Rejection Sampling (Lyubashevsky; EC'11)

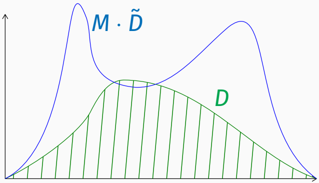

Figure 2: Acceptance zone and sampling domain

$$A_i : (X_1, \ldots, X_i) \mapsto \begin{cases} i & \text{w.p. } \min\left(\frac{D(X_i)}{M \cdot \tilde{D}(X_i)}, 1\right), \\ \perp & \text{otherwise.} \end{cases}$$

Closeness of Imperfect Rejection Sampling

If $\Pr_{x \hookleftarrow D}(D(x) \leq M \cdot \tilde{D}(x)) \geq 1 - \varepsilon$ then the resulting distribution P_{X_J} is such that $\Delta(P_{X_J}, D) \leq \varepsilon$.

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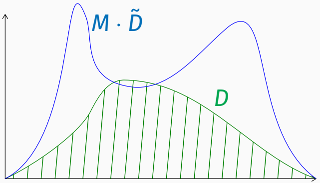

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Let $D, \tilde{D}$ be two probability distributions.

Definition

$$R_\infty(D\|\tilde{D}) = \sup_{x \in \mathrm{Supp}(D)} \frac{D(x)}{\tilde{D}(x)}.$$

Our generalization for any $\varepsilon > 0$:

ε-smooth Rényi divergence

$$R_\infty^\varepsilon(D\|\tilde{D}) = \inf_{\substack{S \\ D(S) \geq 1-\varepsilon}} \sup_{x \in S} \frac{D(x)}{\tilde{D}(x)}.$$

Example: $R_\infty(D^m_{\sigma,\mathbf{c}}\|D^m_\sigma) = +\infty$ whereas $R^\varepsilon_\infty(D^m_{\sigma,\mathbf{c}}\|D^m_\sigma) < +\infty$.

Rényi Divergence

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A family $(A_i : \mathrm{Supp}(\tilde{D})^i \to [i] \cup \{\perp\})_{i \geq 1}$ of randomized algorithms such that $X_J \hookleftarrow D$, where

- $i^* = \min\{i | A_i(X_1, \ldots, X_i) \neq \perp\}$,
- $J = A_{i^*}(X_1, \ldots, X_{i^*})$.

Contribution: Optimality of the standard strategy

Given any strategy $(A_i)_{i \geq 1}$,

$$\mathbb{E}(i^*) \geq R_\infty(D \| \tilde{D}).$$

Reached for $M = R_\infty(D \| \tilde{D})$.

Imperfect RS in terms of Divergence

Contribution: computing the Divergence

$$R_\infty(P_{X_J} \| D) \leq \frac{1}{1-\varepsilon}.$$

Comparisons

- $P_{X_J}(E) \leq D(E) + \varepsilon$ with SD.
- $P_{X_J}(E) \leq \frac{D(E)}{1-\varepsilon} \approx (1+\varepsilon) \cdot D(E)$ with RD.

If Q_s signatures are produced,

$$\varepsilon = \begin{cases} O(2^{-\lambda}) & \text{with SD,} \\ O(1/Q_s) & \text{with RD.} \end{cases}$$

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Minimizing Signature Size

Instantiating Rejection Sampling

Take any discrete P and Q and set

- $\tilde{D}$ distribution of $(\mathbf{z}, \mathbf{c})$ where $\mathbf{y} \hookleftarrow Q$, $\mathbf{c} \hookleftarrow U(\mathcal{C})$ and $\mathbf{z} = \mathbf{y} + \mathbf{Sc}$,
- $D = P \otimes U(\mathcal{C})$,
- $\frac{D(\mathbf{z},\mathbf{c})}{\tilde{D}(\mathbf{z},\mathbf{c})} = \frac{P(\mathbf{z})}{Q(\mathbf{y})}$.

Set $M \geq R_{\infty}^{\varepsilon}(D \| \tilde{D})$ for some $\varepsilon \geq 0$.

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Generic Signature

$\text{Sign}(\mu, \mathbf{A}, \mathbf{S})$:

1: $\mathbf{y} \hookleftarrow Q$
2: $\mathbf{c} \leftarrow H(\mathbf{Ay}, \mu)$
3: $\mathbf{z} \leftarrow \mathbf{y} + \mathbf{Sc}$
4: With probability $\min\left(\frac{P(\mathbf{z})}{M \cdot Q(\mathbf{y})}, 1\right)$ return $(\mathbf{z}, \mathbf{c})$
5: **else** go to Step 1

$\approx \text{Sign}_1(\mu, \mathbf{A}, \mathbf{S})$:

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Can be adapted to work with continuous P and Q.

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Can be adapted to work with continuous P and Q.

Contribution: Understanding how to set the Parameters

Probability Preservation Property

If $\mathcal{A}$ makes Q_s signature queries:

$$\Pr(\mathcal{A}^{\mathsf{Sign}_1} \text{ forges}) \leq \frac{\Pr(\mathcal{A}^{\mathsf{Sign}_2} \text{ forges})}{(1-\varepsilon)^{Q_s}}.$$

Set $\varepsilon = O(1/Q_s)$ (as opposed to $\varepsilon = 2^{-\Omega(\lambda)}$ before).

Multiplicativity

$$R_\infty^\varepsilon(D \| \tilde{D}) \leq \max_{\mathbf{S},\mathbf{c}} R_\infty^\varepsilon(P \| Q_{\mathbf{Sc}}).$$

Set $M \geq \max_{\mathbf{S},\mathbf{c}} R_\infty^\varepsilon(P \| Q_{\mathbf{Sc}})$.

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Our Goal

Let $\varepsilon \geq 0$ and $M > 1$, fixing the runtime.
Let $\Pr_{\mathbf{z} \hookleftarrow P}(\|\mathbf{z}\| > \gamma) = \mathrm{negl}(\lambda)$.
Goal: find P, Q such that $\max_{\mathbf{s},\mathbf{c}} R_\infty^\varepsilon(P \| Q_{\mathbf{sc}}) \leq M$ minimizing γ.

Minimize γ

$\Downarrow$

Cryptanalysis becomes more costly

$\Downarrow$

Smaller parameters overall for the same level of security

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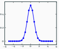
Current choices of Distributions

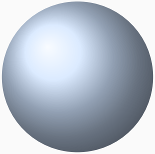
P, Q	Sampling	Rejection	$O(\gamma)_{(\varepsilon=0)}$	$O(\gamma)_{(\varepsilon=\frac{1}{Q_s})}$
$U($$)$	Easy	Deterministic	$\frac{t\sqrt{mm}}{\log M}$	Same
	Cumbersome	Probabilistic	∞	$\frac{t\sqrt{m\log\frac{1}{\varepsilon}}}{\sqrt{\log M}}$

(where $t = \max_{\mathbf{s},\mathbf{c}} \|\mathbf{Sc}\|$)

The first distribution is used in the Dilithium signature scheme.

Our proposal

Use the uniform continuous distribution over



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$U($$)$	Cumbersome	Deterministic	$\frac{tm}{\log M}$	$\frac{t\sqrt{m\log\frac{1}{\varepsilon}}}{\log M}$

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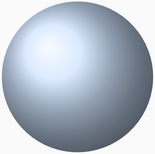

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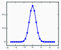	Cumbersome	Probabilistic	∞	$\frac{t\sqrt{m\log\frac{1}{\varepsilon}}}{\sqrt{\log M}}$
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Intuition

Hyperballs versus hypercubes:

- $\{\mathbf{Sc}\} \approx \mathcal{B}_m(t) \cap \mathbb{Z}^m$ and γ is a bound on the Euclidean norm.
- Factor $\sqrt{m}$ gained because of $\|\cdot\| \leq \sqrt{m}\|\cdot\|_\infty$.

Cut and smooth divergence:

- Remove a hyperspherical cap opposed to $\mathbf{Sc}$.
- Volume allowed depends on ε.

Continuous versus discrete hyperballs:

- Easier to study.
- Easier to sample from.

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Optimality

Contribution: Lower bounds on compactness

When $\varepsilon = 0$, for fixed $M > 1$ and any choice of P and Q such that $\max_{\mathbf{s},\mathbf{c}} R_\infty(P \| Q_{\mathbf{sc}}) \leq M$:

$$\gamma \geq \frac{t(m-1)}{\log M}.$$

Intuition of the proof

1. Do the proof for continuous distributions then discretize the result.
2. Model $\{\mathbf{Sc}\}$ as $U(\mathcal{B}_m(t))$.
3. Isotropise P and Q.
4. Deduce a functional inequality on P from the constraint.
5. Solve it.

Bonus: Fixing the Proofs

Claims from the Litterature

Claims

In the standard model for $\Pr_{\mathbf{z} \hookleftarrow P}(\|\mathbf{z}\| \geq \gamma) \leq \mathrm{negl}(\lambda)$ and $t = \max_{\mathbf{S},\mathbf{c}} \|\mathbf{Sc}\|$, the scheme is

- *correct*,
- has expected *number of iterations* M,

thanks to the properties of rejections sampling.

Flaw

- ⚠ D and $\tilde{D}$ are chosen for Sign_1.
- H not a random oracle $\implies$ $\mathbf{z}$ is not distributed following P and expected number of iterations is not M.
- Correctness and runtime analysis relying on this in the litterature are flawed.
- There are examples for which the expected runtime is infinite.

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- Correctness and runtime analysis relying on this in the litterature are flawed.
- There are examples for which the expected runtime is infinite.

Properties

Contribution: Fix the proofs

In the Random Oracle Model, for $\Pr_{\mathbf{z} \hookleftarrow P}(\|\mathbf{z}\| \geq \gamma) \leq \mathrm{negl}(\lambda)$ and $t = \max_{\mathbf{S},\mathbf{c}} \|\mathbf{Sc}\|$, the scheme is

- *correct*,
- *sEU-CMA secure* under the $\mathrm{SIS}_{n,m,2(\gamma+t)}$ assumption,
- and the number of iterations i^* of a call to Sign is such that

$$\Pr(i^* \geq i) \leq \left(1 - \frac{1-\varepsilon}{M}\right)^i + \mathrm{negl}(\lambda).$$

Conclusion and Open Questions

Open questions

1. Concrete instantiation?
2. Efficient sampling from the continuous ball?
3. Totally removing rejection while keeping compactness?
4. Automatisation of rejection-based signature design?

Thank you for your attention!

BLISS

Definitions

Lyubashevsky's Signature Scheme
(Lyubashevsky; AC'09), (Lyubashevsky; EC'11)...

Results on Rejection Sampling

Minimizing Signature Size

BLISS Variant (Ducas et al.; Crypto'13)

Sign$(\mu, \mathbf{A}, \mathbf{S})$:

1: $\mathbf{y} \hookleftarrow Q$
2: $\mathbf{c} \leftarrow H(\mathbf{A}\lceil \mathbf{y} \rfloor, \mu)$
3: $b \hookleftarrow U(\{0,1\})$
4: $\mathbf{z} \leftarrow \mathbf{y} + (-1)^b \mathbf{S}\mathbf{c}$
5: With probability $\min\left(\frac{P(\mathbf{z})}{M \cdot (Q(\mathbf{z}+\mathbf{S}\mathbf{c}) + Q(\mathbf{z}-\mathbf{S}\mathbf{c}))/2}, 1\right)$ return $(\lceil \mathbf{z} \rfloor, \mathbf{c})$
6: **else** go to Step 1

- KeyGen and Verify are adapted to keep correctness and security,

Contribution: Lower bounds

$$\gamma \geq \frac{t\sqrt{m-2}}{\log(M/2)}.$$

- Reached for Gaussians and continuous Hyperball-Uniforms.

BLISS Variant (Ducas et al.; Crypto'13)

Sign$(\mu, \mathbf{A}, \mathbf{S})$:

1: $\mathbf{y} \hookleftarrow Q$
2: $\mathbf{c} \leftarrow H(\mathbf{A}\lceil \mathbf{y} \rfloor, \mu)$
3: $b \hookleftarrow U(\{0, 1\})$
4: $\mathbf{z} \leftarrow \mathbf{y} + (-1)^b \mathbf{Sc}$
5: With probability $\min\left(\frac{P(\mathbf{z})}{M \cdot (Q(\mathbf{z}+\mathbf{Sc}) + Q(\mathbf{z}-\mathbf{Sc}))/2}, 1\right)$ return $(\lceil \mathbf{z} \rfloor, \mathbf{c})$
6: **else** go to Step 1

- KeyGen and Verify are adapted to keep correctness and security,

Contribution: Lower bounds

$$\gamma \geq \frac{t\sqrt{m-2}}{\log(M/2)}.$$

- Reached for Gaussians and continuous Hyperball-Uniforms.