TD 2: Turing Machines and NP-Completeness

1 Turing Machines

Exercise 1. *Stronger Models ?*

1. Show that every bi-infinite tape Turing machine has an equivalent Turing machine.
2. Show that every multiple tapes Turing machine has an equivalent Turing machine.

Exercise 2. *The Halting Problem*

The `HALTING` problem consists in deciding wether a given Turing machine halts on a given word or not.

✎ Show that `HALTING` is undecidable, *i.e.* that there does not exist any Turing machine that accepts the input $< M, w >$ if M halts on w and rejects it otherwise. *(Hint : Adapt the proof for the `ACCEPTATION` problem).*

2 NP-Completeness

Exercise 3. *Restrain Satifiability*

The k-`SATISFIABILITY` problem is a restricted version of `SATISFIABILITY` in which all instances have exactly k literals per clause.

✎ Show that `3-SATISFIABILITY` is NP-Complete. *(Hint : Do a reduction from `SATISFIABILITY`).*

Culture : `2-SATISFIABILITY` is not NP-Complete (we know a polynomial algorithm).

Exercise 4. *Graphs' Subsets*

Let $G = (V, E)$ be a graph. We define :

— An *independent set* is a subset $V' \subseteq V$ such that for all $u, v \in V'$, $\{u, v\}$ is *not* in E.
— A *vertex cover* is a subset $V' \subseteq V$ such that for all $\{u, v\} \in E$, $u \in V'$ or $v \in V'$.
— A *clique* is a subset $V' \subseteq V$ such that for all $u, v \in V'$, $\{u, v\}$ is in E.

1. For any graph $G = (V, E)$ and any subset $V' \subseteq V$, show that the following statements are equivalent :
   (i) V' is a vertex cover for G.
   (ii) $V \backslash V'$ is an independent set for G.
   (iii) $V \backslash V'$ is a cliqque in the complement G^c of G where $G^c = (V, E^c)$ with $E^c = \{\{u, v\}, u, v \in V \text{ and } \{u, v) \notin E\}$.

Let $G = (V, E)$ be a graph and k be an integer. We define the following problem on graphs :

— `INDEPENDANT-SET` is the problem to know whether there exists an independent set of size greater than or equal to k in G or not.
— `VERTEX-COVER` is the problem to know whether there exists a vertex cover of size less than or equal to k in G or not.

— CLIQUE is the problem to know whether there exists a clique of size greater than or equal to k in G or not.

2. Show that VERTEX-COVER is NP-Complete. *(Hint : Do a reduction from 3-SATISFIABILITY).*

3. Deduce that INDEPENDENT-SET and CLIQUE are NP-Complete.

Culture : EDGE-COVER is like VERTEX-COVER but we look for a subset $E' \subseteq E$ such that every vertex of V is incident to at least one vertex in E'. Yet it is not NP-Complete (we know a polynomial algorithm).

Exercise 5. *A Difficult Reduction*

Let $G = (V, E)$ be a graph. A *Hamiltonian circuit* in G is a cycle in the graph going through each vertex exactly once. The HAMILTONIAN-CIRCUIT problem consists in deciding wether a given graph contains a Hamiltonian circuit or not.

✎ Show that HAMILTONIAN-CIRCUIT is NP-Complete. *(Hint : Do a reduction from VERTEX-COVER).*

Culture : EULERIAN CIRCUIT is like HAMILTONIAN-CIRCUIT but we look for a cycle that traverses every edge exactly once. Yet it is not NP-Complete (we know a polynomial algorithm).