Cyclic Proofs and jumping automata

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Lyon

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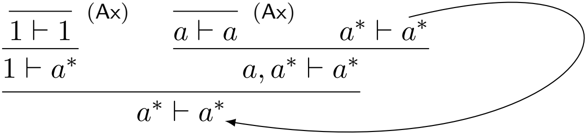
Cyclic proofs

Regular expressions

$$e, f := 1 \mid a \in A \mid e \cdot f \mid e + f \mid e^*$$

<u>Context:</u> Cyclic proofs for inclusion of expressions [Das, Pous '17]

- Infinite proof trees, with root of the form $e \vdash f$.



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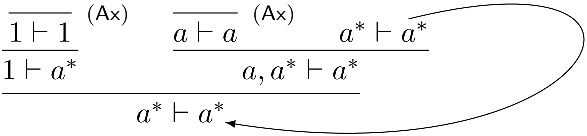

- Validity condition on infinite branches

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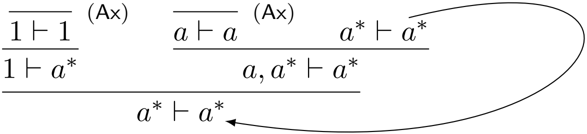

- Validity condition on infinite branches
- $\exists$ proof of $e \vdash f \Leftrightarrow L(e) \subseteq L(f)$.

Computational interpretation

Proof of $e \vdash f$

$\rightsquigarrow$ Computational interpretation

Program with input from e and output in f.

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Several programs of the same type

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$a \vdash a + a$

in_l or in_r

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Curry-Howard isomorphism, typed programming,. . .

Well-understood for finite proofs, active field for infinite proofs.

This work

- Boolean type $2 = 1 + 1$
- Add *structural* rules corresponding to simple natural programs
- Study the expressive power of regular proofs (finite graphs)
- Focus on proofs for languages:

$$\text{Proof } \pi \text{ of } A^* \vdash 2 \rightsquigarrow \text{Language } L(\pi) \subseteq A^*$$

Proof system

Expressions $e := A \mid A^*$

Sequents $E, F = e_1, e_2, \dots, e_n$

Proof system with extra rules for basic data manipulation:

$$\frac{}{\vdash 2}\ (\mathbf{tt}) \qquad \frac{}{\vdash 2}\ (\mathbf{ff})$$

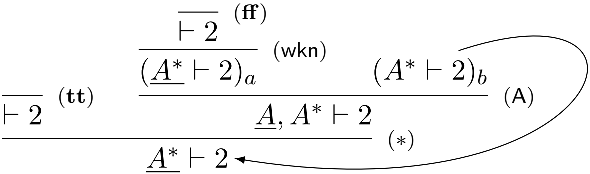
$$\frac{E, F \vdash 2}{E, \underline{e}, F \vdash 2}\ (\mathsf{wkn}) \qquad \frac{E, \underline{e, e}, F \vdash 2}{E, \underline{e}, F \vdash 2}\ (\mathsf{ctr})$$

$$\frac{(E, F \vdash 2)_{\underline{a \in A}}}{E, \underline{A}, F \vdash 2}\ (\mathsf{A}) \qquad \frac{E, F \vdash 2 \qquad E, \underline{A, A^*}, F \vdash 2}{E, \underline{A^*}, F \vdash 2}\ (*)$$

Proofs as language acceptors

What are the languages computed by cyclic proofs ?

Example on alphabet $\{a, b\}$: b^*



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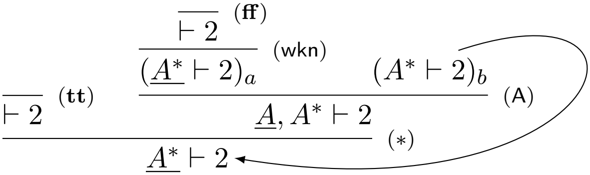
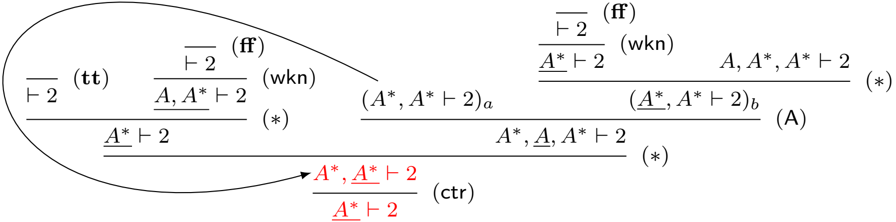


Lemma

Without contraction, the system captures exactly regular languages.

With contractions: what class of language?

Example on alphabet $\{a, b\}$: $a^n b^n$



1st step : create a copy of the input and delete the first a's.

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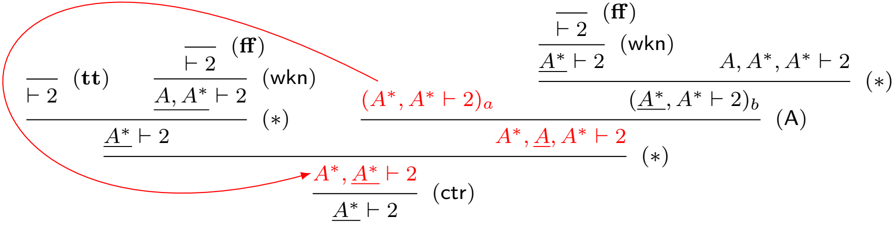

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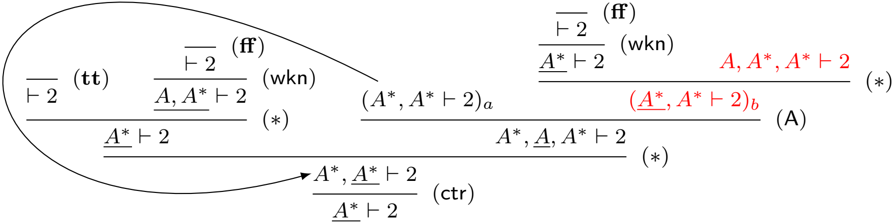

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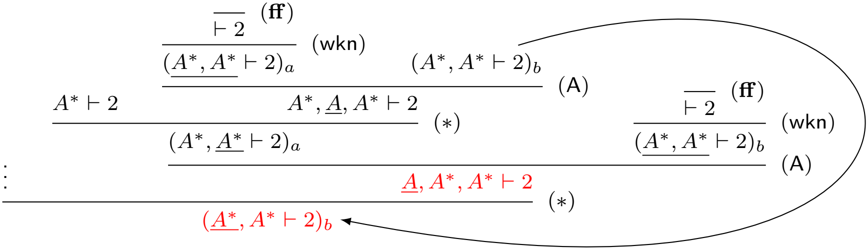

2nd step : check that for each b of the second copy we have a a in the first one.

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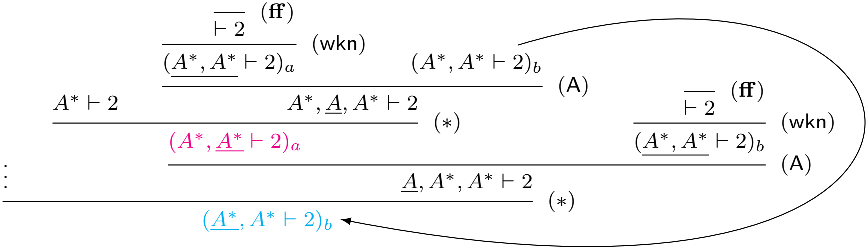

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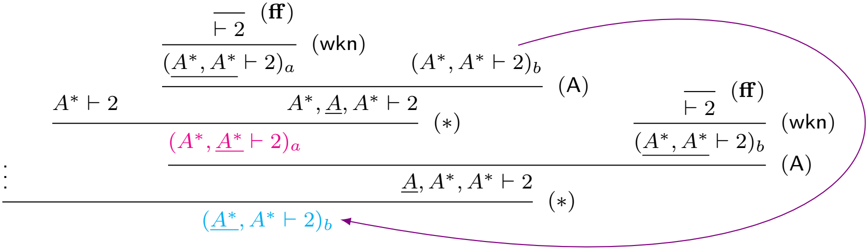

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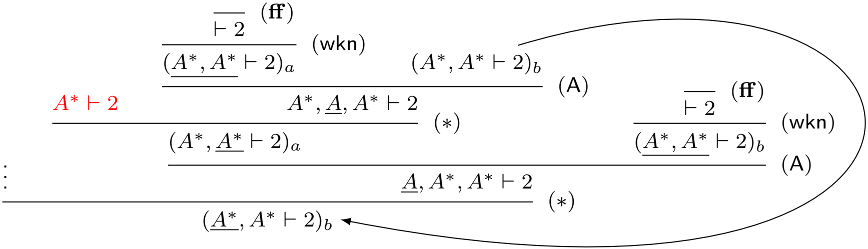

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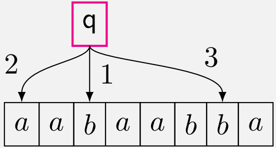
Example on alphabet $\{a, b\}$: $a^n b^n c^n$

With contractions: a new automaton model

Jumping Multihead Automata

A JMA is an automaton with k reading heads.

Transitions: $Q \times A^k \to Q \times \{$▶|, ⁙, $J_1, \ldots, J_k\}^k$



- ▶| : advance one step
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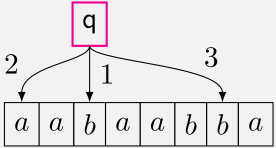

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+ *Equivalent of the validity criterion*

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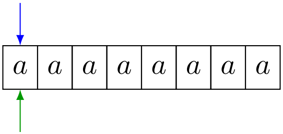
Example: $\{a^{2^n} \mid n \in \mathbb{N}\}$ is accepted by a 2-head JMA.



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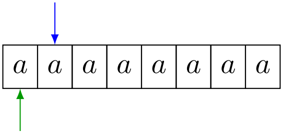

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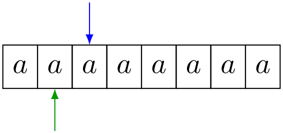

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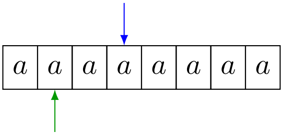

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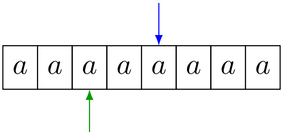

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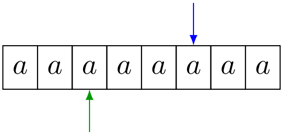

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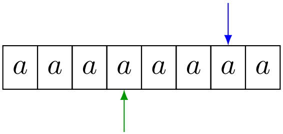

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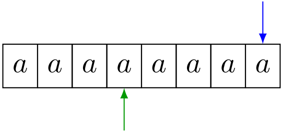

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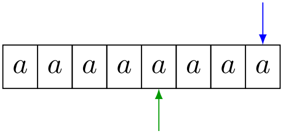

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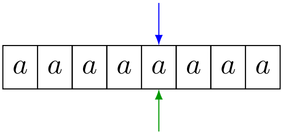

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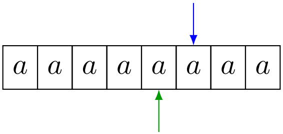

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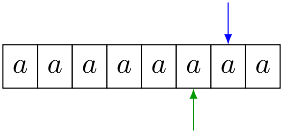

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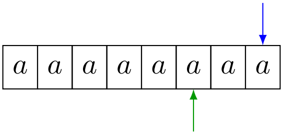

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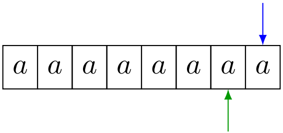

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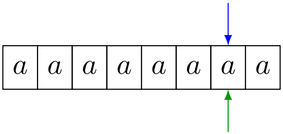

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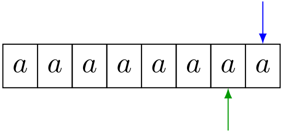

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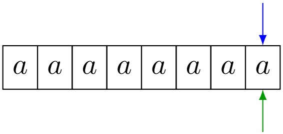

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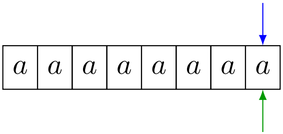

ACCEPT

Expressive power of JMA

Comparison with Multihead Automata in Litterature:

[Holzer, Kutrib, Malcher 2008]

1-way Multihead

2-way Multihead

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Main result

Theorem

Cyclic proofs and JMA recognize the same class of languages.

States of the automaton	$\sim$	Positions in the proof tree
Accepting / Rejecting state	$\sim$	True / False axiom
Multiple heads	$\sim$	Multiple copies of A^*
Reading a letter	$\sim$	Applying $*$ and (A) rules

What next ?

- Add the cut rule
- Corresponds to composition of functions
- Sequents $(1^*)^k \vdash 1^*$: functions $\mathbb{N}^k \to \mathbb{N}$

Work in progress:

No contraction	=	Primitive Recursive
With contraction	=	System T

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Thank you for your attention !

[Denis Kuperberg, Laureline Pinault and Damien Pous, FSTTCS 19]