CENTRAL CHARACTERS FOR SMOOTH IRREDUCIBLE MODULAR REPRESENTATIONS OF GL$_2$(Q$_p$)

by

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To Francesco Baldassarri, on the occasion of his 60th birthday

Abstract. — We prove that every smooth irreducible $\mathbb{F}_p$-linear representation of GL$_2$(Q$_p$) admits a central character.

Introduction

Let $\Pi$ be a representation of GL$_2$(Q$_p$). We say that $\Pi$ is smooth, if the stabilizer of any $v \in \Pi$ is an open subgroup of GL$_2$(Q$_p$). We say that $\Pi$ admits a central character, if every $z \in Z$(GL$_2$(Q$_p$)), the center of GL$_2$(Q$_p$), acts on $\Pi$ by a scalar. The smooth irreducible representations of GL$_2$(Q$_p$) over an algebraically closed field of characteristic $p$, admitting a central character, have been studied by Barthel–Livné in [BL94, BL95] and by Breuil in [Bre03]. The purpose of this note is to prove the following theorem.

Theorem A. — If $\Pi$ is a smooth irreducible $\mathbb{F}_p$-linear representation of GL$_2$(Q$_p$), then $\Pi$ admits a central character.

The idea of the proof of theorem A is as follows. If $\Pi$ does not admit a central character, and if $f = (\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix})$, then for any nonzero polynomial $Q(X) \in \mathbb{F}_p[X]$, the map $Q(f) : \Pi \to \Pi$ is bijective, so that $\Pi$ has the structure of a $\mathbb{F}_p(X)$-vector space. The representation $\Pi$ is therefore a smooth irreducible $\mathbb{F}_p(X)$-linear representation of GL$_2$(Q$_p$), which now admits a central character, since $f$ acts by multiplication by $X$. It remains to apply Barthel–Livné and Breuil’s classification, which gives the structure of the components of $\Pi$ after extending scalars to a finite extension $K$ of $\mathbb{F}_p(X)$. A corollary of this classification is that these components are all “defined” over a subring $R$ of $K$, where $R$ is a finitely generated $\mathbb{F}_p$-algebra. This can be used to show that $\Pi$ is not of finite length, a contradiction.

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Note that it is customary to ask that smooth irreducible representations of $\text{GL}_2(\mathbb{Q}_p)$ also be admissible (meaning that $\Pi^U$ is finite-dimensional for every open compact subgroup $U$ of $G$). A corollary of Barthel–Livné and Breuil’s classification is that every smooth irreducible $\mathbb{F}_p$-linear representation of $\text{GL}_2(\mathbb{Q}_p)$ that admits a central character is admissible, and hence theorem A implies that every smooth irreducible $\mathbb{F}_p$-linear representation of $\text{GL}_2(\mathbb{Q}_p)$ is admissible. In particular, such a representation also satisfies Schur’s lemma: every $\text{GL}_2(\mathbb{Q}_p)$-equivariant map is a scalar. Our theorem A can also be seen as a special case of Schur’s lemma, since $(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix})$ is a $\text{GL}_2(\mathbb{Q}_p)$-equivariant map.

There are (at least) two standard ways of proving Schur’s lemma: one way uses admissibility, and the other works for smooth irreducible $E$-linear representations of $\text{GL}_2(\mathbb{Q}_p)$, but only if $E$ is uncountable (see proposition 2.11 of [BZ76]). In order to prove theorem A, we cannot simply extend scalars to an uncountable extension of $\mathbb{F}_p$, as we do not know whether the resulting representation will still be irreducible.

We finish this introduction by pointing out that a few years ago, Henniart had sketched a different (and more complicated) argument for the proof of theorem A.

1. Barthel–Livné and Breuil’s classification

Let $E$ be a field of characteristic $p$. In this section, we recall the explicit classification of smooth irreducible $E$-linear representations of $\text{GL}_2(\mathbb{Q}_p)$, admitting a central character.

We denote the center of $\text{GL}_2(\mathbb{Q}_p)$ by $Z$. If $r \geq 0$, then $\text{Sym}^r E^2$ is a representation of $\text{GL}_2(\mathbb{F}_p)$ which gives rise, by inflation, to a representation of $\text{GL}_2(\mathbb{Z}_p)$. We extend it to $\text{GL}_2(\mathbb{Z}_p)Z$ by letting $(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix})$ act trivially. Consider the representation

$$\text{ind}_{\text{GL}_2(\mathbb{Z}_p)Z}^{\text{GL}_2(\mathbb{Q}_p)} \text{Sym}^r E^2.$$ 

The Hecke algebra

$$\text{End}_{E[\text{GL}_2(\mathbb{Q}_p)]} \left( \text{ind}_{\text{GL}_2(\mathbb{Z}_p)Z}^{\text{GL}_2(\mathbb{Q}_p)} \text{Sym}^r E^2 \right)$$

is isomorphic to $E[T]$ where $T$ is a Hecke operator, which corresponds to the double class $\text{GL}_2(\mathbb{Z}_p)Z \cdot (\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}) \cdot \text{GL}_2(\mathbb{Z}_p)$. If $\chi : \mathbb{Q}_p^\times \to E^\times$ is a smooth character, and if $\lambda \in E$, then let

$$\pi(r, \lambda, \chi) = \frac{\text{ind}_{\text{GL}_2(\mathbb{Z}_p)Z}^{\text{GL}_2(\mathbb{Q}_p)} \text{Sym}^r E^2}{T - \lambda} \otimes (\chi \circ \text{det}).$$

This is a smooth representation of $\text{GL}_2(\mathbb{Q}_p)$, with central character $\omega^r \chi^2$ (where $\omega : \mathbb{Q}_p^\times \to \mathbb{F}_p^\times$ is given by $p^n x_0 \mapsto x_0$, with $x_0 \in \mathbb{Z}_p^\times$). Let $\mu_\lambda : \mathbb{Q}_p^\times \to E^\times$ be given by $\mu_\lambda|_{\mathbb{Z}_p^\times} = 1$, and $\mu_\lambda(p) = \lambda$. If $\lambda = \pm 1$, then we have two exact sequences:

$$0 \to \text{Sp}_E \otimes (\chi \mu_\lambda \circ \text{det}) \to \pi(0, \lambda, \chi) \to \chi \mu_\lambda \circ \text{det} \to 0,$$

$$0 \to \chi \mu_\lambda \circ \text{det} \to \pi(p - 1, \lambda, \chi) \to \text{Sp}_E \otimes (\chi \mu_\lambda \circ \text{det}) \to 0,$$
where the representation $\text{Sp}_E$ is the “special” representation with coefficients in $E$.

**Theorem 1.1.** — If $E$ is algebraically closed, then the smooth irreducible $E$-linear representations of $\text{GL}_2(\mathbb{Q}_p)$, admitting a central character, are as follows:

1. $\chi \circ \text{det}$;
2. $\text{Sp}_E \otimes (\chi \circ \text{det})$;
3. $\pi(r, \lambda, \chi)$, where $r \in \{0, \ldots, p - 1\}$ and $(r, \lambda) \notin \{(0, \pm 1), (p - 1, \pm 1)\}$.

This theorem is proved in [BL95] and [BL94], which treat the case $\lambda \neq 0$, and in [Bre03], which treats the case $\lambda = 0$.

We now explain what happens if $E$ is not algebraically closed.

**Proposition 1.2.** — If $\Pi$ is a smooth irreducible $E$-linear representation of $\text{GL}_2(\mathbb{Q}_p)$, admitting a central character, then there exists a finite extension $K/E$ such that $(\Pi \otimes_E E^K)_{ss}$ is a direct sum of $K$-linear representations of the type described in theorem 1.1.

**Proof.** — Barthel and Livné’s methods show (as is observed in §5.3 of [Paś10]) that $\Pi$ is a quotient of

$$\Sigma = \frac{\text{ind}_{\text{GL}_2(\mathbb{Z}_p)Z}^{\text{GL}_2(\mathbb{Q}_p)Z} \text{Sym}^r E^2}{P(T)} \otimes (\chi \circ \text{det}),$$

for some integer $r \in \{0, \ldots, p - 1\}$, character $\chi : \mathbb{Q}_p^\times \to E^\times$, and polynomial $P(Y) \in E[Y]$. Let $K$ be a splitting field of $P(Y)$, write $P(Y) = (Y - \lambda_1) \cdots (Y - \lambda_d)$, and let $P_i(Y) = (Y - \lambda_1) \cdots (Y - \lambda_i)$ for $i = 0, \ldots, d$. The representations $P_{i-1}(T)\Sigma/P_i(T)\Sigma$ are then subquotients of the $\pi(r, \lambda, \chi)$, for $i = 1, \ldots, d$.

We finish this section by recalling that if $\lambda \neq 0$, then the representations $\pi(r, \lambda, \chi)$ are parabolic inductions (when $\lambda = 0$, they are called supersingular). Let $B_2(\mathbb{Q}_p)$ be the upper triangular Borel subgroup of $\text{GL}_2(\mathbb{Q}_p)$, let $\chi_1$ and $\chi_2 : \mathbb{Q}_p^\times \to E^\times$ be two smooth characters, and consider the parabolic induction $\text{ind}_{B_2(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)}(\chi_1 \otimes \chi_2)$. The following result is proved in [BL94] and [BL95].

**Theorem 1.3.** — If $\lambda \in E \setminus \{0; \pm 1\}$, and if $r \in \{0, \ldots, p - 1\}$, then $\pi(r, \lambda, \chi)$ is isomorphic to $\text{ind}_{B_2(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)}(\chi_{\mu_1/\lambda} \otimes \chi_{\omega^r/\mu_\lambda})$.

### 2. Proof of the theorem

We now give the proof of theorem A. Let $\Pi$ be a smooth irreducible $\overline{\mathbb{F}}_p$-linear representation of $\text{GL}_2(\mathbb{Q}_p)$. We have $\Pi^{(1 + p\mathbb{Z}_p) \cdot \text{Id}} \neq 0$ (since a $p$-group acting on a $\mathbb{F}_p$-vector space always has nontrivial fixed points), so that if $\Pi$ is irreducible, then $(1 + p\mathbb{Z}_p) \cdot \text{Id}$
acts trivially on \( \Pi \). If \( g \in \mathbb{Z}_p^* \cdot \text{Id} \), then \( g^{p-1} = \text{Id} \) on \( \Pi \), so that \( \Pi = \oplus_{\omega \in \mathbb{F}_p} \Pi^{g=\omega \cdot \text{Id}} \). Since \( \Pi \) is irreducible, this implies that the elements of \( \mathbb{Z}_p^* \cdot \text{Id} \) act by scalars.

If \( f = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \), then for any nonzero polynomial \( Q(X) \in \mathbb{F}_p[X] \), the kernel and image of the map \( Q(f) : \Pi \to \Pi \) are subrepresentations of \( \Pi \). If \( Q(f) = 0 \) on a nontrivial subspace of \( \Pi \), then \( f \) admits an eigenvector for an eigenvalue \( \lambda \in \mathbb{F}_p^\times \). This implies that \( \Pi = \Pi^{f=\lambda \cdot \text{Id}} \), so that \( \Pi \) does admit a central character. If this is not the case, then \( Q(f) \) is bijective for every nonzero polynomial \( Q(X) \in \mathbb{F}_p[X] \), so that \( \Pi \) has the structure of a \( \mathbb{F}_p(X) \)-vector space, and is a \( \mathbb{F}_p(X) \)-linear smooth irreducible representation of \( \text{GL}_2(\mathbb{Q}_p) \), admitting a central character.

Let \( E = \overline{\mathbb{F}}_p(X) \). Proposition 1.2 gives us a finite extension \( K \) of \( E \), such that \( (\Pi \otimes_E K)^{ss} \) is a direct sum of \( K \)-linear representations of the type described in theorem 1.1. The \( \mathbb{F}_p \)-linear representation underlying \( (\Pi \otimes_E K)^{ss} \) is isomorphic to \( \Pi^{[K:E]} \), and hence of length \([K:E]\). We now prove that none of the \( K \)-linear representations of the type described in theorem 1.1 are of finite length, when viewed as \( \mathbb{F}_p \)-linear representations.

Let \( \Sigma \) be one such representation, and let \( \lambda \in K \) be the corresponding Hecke eigenvalue.

We now construct a subring \( R \) of \( K \), which is a finitely generated \( \mathbb{F}_p \)-algebra, and an \( R \)-linear representation \( \Sigma_R \) of \( \text{GL}_2(\mathbb{Q}_p) \), such that \( \Sigma = \Sigma_R \otimes_R K \).

If \( \lambda \in \mathbb{F}_p \), then theorem 1.1 shows that

\[
\Sigma = \frac{\text{ind}_{\text{GL}_2(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Z}_p)}}{T - \lambda} \mathbb{F}_p^2 \otimes_{\mathbb{F}_p} K(\chi \circ \det), \text{ or } \text{Sp}_{\mathbb{F}_p} \otimes_{\mathbb{F}_p} K(\chi \circ \det), \text{ or } K(\chi \circ \det).
\]

We can then take \( R = \mathbb{F}_p(\chi(p))^{\pm 1} \), and \( \Sigma_R = (\text{ind}_{\text{GL}_2(\mathbb{Z}_p)}^{\text{GL}_2(\mathbb{Q}_p)}) \mathbb{F}_p^2/(T - \lambda)) \otimes_{\mathbb{F}_p} R(\chi \circ \det), \text{ or } \text{Sp}_{\mathbb{F}_p} \otimes_{\mathbb{F}_p} R(\chi \circ \det), \text{ or } R(\chi \circ \det), \) respectively.

If \( \lambda \notin \overline{\mathbb{F}}_p \), then by theorem 1.3, we have

\[
\Sigma = \text{ind}_{\text{B}_2(\mathbb{Q}_p)}^{\text{GL}_2(\mathbb{Q}_p)}(\chi \mu_1/\lambda, \chi \omega^r \mu_\lambda).
\]

We can take \( R = \mathbb{F}_p[\lambda^{\pm 1}, \chi(p)^{\pm 1}] \), and let \( \Sigma_R \) be the set of functions \( f \in \Sigma \) with values in \( R \).

In the first case, \( \Sigma_R \) is a free \( R \)-module, while in the second case, \( \Sigma_R \) is isomorphic as an \( R \)-module to \( C^\infty(\mathbb{P}^1(\mathbb{Q}_p), R) \) and hence also free. In either case, if \( f \in R \) is nonzero and not a unit and \( j \in \mathbb{Z} \), then \( f^{j+1} \cdot \Sigma_R \) is a proper \( \mathbb{F}_p \)-linear subrepresentation of \( f^j \cdot \Sigma_R \), so that the underlying \( \mathbb{F}_p \)-linear representation of \( \Sigma_R \) is not of finite length. Since \( \Sigma_R \subset \Sigma \), the underlying \( \mathbb{F}_p \)-linear representation of \( \Sigma \) is not of finite length, which is a contradiction. This finishes the proof of theorem A.

References


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