CENTRAL CHARACTERS FOR SMOOTH IRREDUCIBLE MODULAR REPRESENTATIONS OF $\mathrm{GL}_2(\mathbf{Q}_p)$

by

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To Francesco Baldassarri, on the occasion of his 60th birthday

Abstract. — We prove that every smooth irreducible $\overline{\mathbf{F}}_p$-linear representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ admits a central character.

Introduction

Let Π be a representation of $\mathrm{GL}_2(\mathbf{Q}_p)$. We say that Π is smooth, if the stabilizer of any $v \in \Pi$ is an open subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$. We say that Π admits a central character, if every $z \in \mathrm{Z}(\mathrm{GL}_2(\mathbf{Q}_p))$, the center of $\mathrm{GL}_2(\mathbf{Q}_p)$, acts on Π by a scalar. The smooth irreducible representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ over an algebraically closed field of characteristic p, admitting a central character, have been studied by Barthel–Livné in [**BL94, BL95**] and by Breuil in [**Bre03**]. The purpose of this note is to prove the following theorem.

Theorem A. — *If* Π *is a smooth irreducible* $\overline{\mathbf{F}}_p$*-linear representation of* $\mathrm{GL}_2(\mathbf{Q}_p)$*, then* Π *admits a central character.*

The idea of the proof of theorem A is as follows. If Π does not admit a central character, and if $f = \left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$, then for any nonzero polynomial $Q(X) \in \overline{\mathbf{F}}_p[X]$, the map $Q(f) : \Pi \to \Pi$ is bijective, so that Π has the structure of a $\overline{\mathbf{F}}_p(X)$-vector space. The representation Π is therefore a smooth irreducible $\overline{\mathbf{F}}_p(X)$-linear representation of $\mathrm{GL}_2(\mathbf{Q}_p)$, which now admits a central character, since f acts by multiplication by X. It remains to apply Barthel–Livné and Breuil's classification, which gives the structure of the components of Π after extending scalars to a finite extension K of $\overline{\mathbf{F}}_p(X)$. A corollary of this classification is that these components are all "defined" over a subring R of K, where R is a finitely generated $\overline{\mathbf{F}}_p$-algebra. This can be used to show that Π is not of finite length, a contradiction.

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Note that it is customary to ask that smooth irreducible representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ also be admissible (meaning that Π^U is finite-dimensional for every open compact subgroup U of G). A corollary of Barthel–Livné and Breuil's classification is that every smooth irreducible $\overline{\mathbf{F}}_p$-linear representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ that admits a central character is admissible, and hence theorem A implies that every smooth irreducible $\overline{\mathbf{F}}_p$-linear representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ is admissible. In particular, such a representation also satisfies Schur's lemma: every $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant map is a scalar. Our theorem A can also be seen as a special case of Schur's lemma, since $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ is a $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant map.

There are (at least) two standard ways of proving Schur's lemma: one way uses admissibility, and the other works for smooth irreducible E-linear representations of $\mathrm{GL}_2(\mathbf{Q}_p)$, but only if E is uncountable (see proposition 2.11 of [**BZ76**]). In order to prove theorem A, we cannot simply extend scalars to an uncountable extension of $\overline{\mathbf{F}}_p$, as we do not know whether the resulting representation will still be irreducible.

We finish this introduction by pointing out that a few years ago, Henniart had sketched a different (and more complicated) argument for the proof of theorem A.

1. Barthel–Livné and Breuil's classification

Let E be a field of characteristic p. In this section, we recall the explicit classification of smooth irreducible E-linear representations of $\mathrm{GL}_2(\mathbf{Q}_p)$, admitting a central character.

We denote the center of $\mathrm{GL}_2(\mathbf{Q}_p)$ by Z. If $r \geqslant 0$, then $\mathrm{Sym}^r E^2$ is a representation of $\mathrm{GL}_2(\mathbf{F}_p)$ which gives rise, by inflation, to a representation of $\mathrm{GL}_2(\mathbf{Z}_p)$. We extend it to $\mathrm{GL}_2(\mathbf{Z}_p)\,\mathrm{Z}$ by letting $\left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$ act trivially. Consider the representation

$$\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\,\mathrm{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^r E^2.$$

The Hecke algebra

$$\mathrm{End}_{E[\mathrm{GL}_2(\mathbf{Q}_p)]}\left(\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\,\mathrm{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^r E^2\right)$$

is isomorphic to $E[T]$ where T is a Hecke operator, which corresponds to the double class $\mathrm{GL}_2(\mathbf{Z}_p)\,\mathrm{Z}\cdot\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\cdot\mathrm{GL}_2(\mathbf{Z}_p)$. If $\chi : \mathbf{Q}_p^\times \to E^\times$ is a smooth character, and if $\lambda \in E$, then let

$$\pi(r,\lambda,\chi) = \frac{\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\,\mathrm{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^r E^2}{T-\lambda} \otimes (\chi \circ \det).$$

This is a smooth representation of $\mathrm{GL}_2(\mathbf{Q}_p)$, with central character $\omega^r\chi^2$ (where $\omega : \mathbf{Q}_p^\times \to \mathbf{F}_p^\times$ is given by $p^n x_0 \mapsto \overline{x}_0$, with $x_0 \in \mathbf{Z}_p^\times$). Let $\mu_\lambda : \mathbf{Q}_p^\times \to E^\times$ be given by $\mu_\lambda|_{\mathbf{Z}_p^\times} = 1$, and $\mu_\lambda(p) = \lambda$. If $\lambda = \pm 1$, then we have two exact sequences:

$$0 \to \mathrm{Sp}_E \otimes (\chi\mu_\lambda \circ \det) \to \pi(0,\lambda,\chi) \to \chi\mu_\lambda \circ \det \to 0,$$

$$0 \to \chi\mu_\lambda \circ \det \to \pi(p-1,\lambda,\chi) \to \mathrm{Sp}_E \otimes (\chi\mu_\lambda \circ \det) \to 0,$$

where the representation Sp_E is the "special" representation with coefficients in E.

Theorem 1.1. — *If E is algebraically closed, then the smooth irreducible E-linear representations of $\mathrm{GL}_2(\mathbf{Q}_p)$, admitting a central character, are as follows:*

1. $\chi \circ \det$;
2. $\mathrm{Sp}_E \otimes (\chi \circ \det)$;
3. $\pi(r, \lambda, \chi)$, *where* $r \in \{0, \ldots, p-1\}$ *and* $(r, \lambda) \notin \{(0, \pm 1), (p-1, \pm 1)\}$.

This theorem is proved in [**BL95**] and [**BL94**], which treat the case $\lambda \neq 0$, and in [**Bre03**], which treats the case $\lambda = 0$.

We now explain what happens if E is not algebraically closed.

Proposition 1.2. — *If Π is a smooth irreducible E-linear representation of $\mathrm{GL}_2(\mathbf{Q}_p)$, admitting a central character, then there exists a finite extension K/E such that $(\Pi \otimes_E K)^{\mathrm{ss}}$ is a direct sum of K-linear representations of the type described in theorem 1.1.*

Proof. — Barthel and Livné's methods show (as is observed in §5.3 of [**Paš10**]) that Π is a quotient of

$$\Sigma = \frac{\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathrm{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^r E^2}{P(T)} \otimes (\chi \circ \det),$$

for some integer $r \in \{0, \ldots, p-1\}$, character $\chi : \mathbf{Q}_p^\times \to E^\times$, and polynomial $P(Y) \in E[Y]$. Let K be a splitting field of $P(Y)$, write $P(Y) = (Y - \lambda_1) \cdots (Y - \lambda_d)$, and let $P_i(Y) = (Y - \lambda_1) \cdots (Y - \lambda_i)$ for $i = 0, \ldots, d$. The representations $P_{i-1}(T)\Sigma / P_i(T)\Sigma$ are then subquotients of the $\pi(r, \lambda_i, \chi)$, for $i = 1, \ldots, d$. □

We finish this section by recalling that if $\lambda \neq 0$, then the representations $\pi(r, \lambda, \chi)$ are parabolic inductions (when $\lambda = 0$, they are called supersingular). Let $\mathrm{B}_2(\mathbf{Q}_p)$ be the upper triangular Borel subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$, let χ_1 and $\chi_2 : \mathbf{Q}_p^\times \to E^\times$ be two smooth characters, and consider the parabolic induction $\mathrm{ind}_{\mathrm{B}_2(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)}(\chi_1 \otimes \chi_2)$. The following result is proved in [**BL94**] and [**BL95**].

Theorem 1.3. — *If $\lambda \in E \setminus \{0; \pm 1\}$, and if $r \in \{0, \ldots, p-1\}$, then $\pi(r, \lambda, \chi)$ is isomorphic to* $\mathrm{ind}_{\mathrm{B}_2(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)}(\chi \mu_{1/\lambda} \otimes \chi \omega^r \mu_\lambda)$.

2. Proof of the theorem

We now give the proof of theorem A. Let Π be a smooth irreducible $\overline{\mathbf{F}}_p$-linear representation of $\mathrm{GL}_2(\mathbf{Q}_p)$. We have $\Pi^{(1+p\mathbf{Z}_p)\cdot \mathrm{Id}} \neq 0$ (since a p-group acting on a $\mathbf{F}_p$-vector space always has nontrivial fixed points), so that if Π is irreducible, then $(1 + p\mathbf{Z}_p) \cdot \mathrm{Id}$

acts trivially on Π. If $g \in \mathbf{Z}_p^\times \cdot \mathrm{Id}$, then $g^{p-1} = \mathrm{Id}$ on Π, so that $\Pi = \oplus_{\omega \in \mathbf{F}_p^\times} \Pi^{g=\omega\cdot\mathrm{Id}}$. Since Π is irreducible, this implies that the elements of $\mathbf{Z}_p^\times \cdot \mathrm{Id}$ act by scalars.

If $f = \left(\begin{smallmatrix} p & 0 \\ 0 & p \end{smallmatrix}\right)$, then for any nonzero polynomial $Q(X) \in \overline{\mathbf{F}}_p[X]$, the kernel and image of the map $Q(f) : \Pi \to \Pi$ are subrepresentations of Π. If $Q(f) = 0$ on a nontrivial subspace of Π, then f admits an eigenvector for an eigenvalue $\lambda \in \overline{\mathbf{F}}_p^\times$. This implies that $\Pi = \Pi^{f=\lambda\cdot\mathrm{Id}}$, so that Π does admit a central character. If this is not the case, then $Q(f)$ is bijective for every nonzero polynomial $Q(X) \in \overline{\mathbf{F}}_p[X]$, so that Π has the structure of a $\overline{\mathbf{F}}_p(X)$-vector space, and is a $\overline{\mathbf{F}}_p(X)$-linear smooth irreducible representation of $\mathrm{GL}_2(\mathbf{Q}_p)$, admitting a central character.

Let $E = \overline{\mathbf{F}}_p(X)$. Proposition 1.2 gives us a finite extension K of E, such that $(\Pi \otimes_E K)^{\mathrm{ss}}$ is a direct sum of K-linear representations of the type described in theorem 1.1. The $\overline{\mathbf{F}}_p$-linear representation underlying $(\Pi \otimes_E K)^{\mathrm{ss}}$ is isomorphic to $\Pi^{[K:E]}$, and hence of length $[K : E]$. We now prove that none of the K-linear representations of the type described in theorem 1.1 are of finite length, when viewed as $\overline{\mathbf{F}}_p$-linear representations.

Let Σ be one such representation, and let $\lambda \in K$ be the corresponding Hecke eigenvalue. We now construct a subring R of K, which is a finitely generated $\overline{\mathbf{F}}_p$-algebra, and an R-linear representation Σ_R of $\mathrm{GL}_2(\mathbf{Q}_p)$, such that $\Sigma = \Sigma_R \otimes_R K$.

If $\lambda \in \overline{\mathbf{F}}_p$, then theorem 1.1 shows that

$$\Sigma = \frac{\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathrm{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^r\, \overline{\mathbf{F}}_p^2}{T-\lambda} \otimes_{\overline{\mathbf{F}}_p} K(\chi \circ \det), \text{ or } \mathrm{Sp}_{\overline{\mathbf{F}}_p} \otimes_{\overline{\mathbf{F}}_p} K(\chi \circ \det), \text{ or } K(\chi \circ \det).$$

We can then take $R = \overline{\mathbf{F}}_p[\chi(p)^{\pm 1}]$, and $\Sigma_R = (\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathrm{Z}}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^r\, \overline{\mathbf{F}}_p^2/(T-\lambda)) \otimes_{\overline{\mathbf{F}}_p} R(\chi \circ \det)$, or $\mathrm{Sp}_{\overline{\mathbf{F}}_p} \otimes_{\overline{\mathbf{F}}_p} R(\chi \circ \det)$, or $R(\chi \circ \det)$, respectively.

If $\lambda \notin \overline{\mathbf{F}}_p$, then by theorem 1.3, we have

$$\Sigma = \mathrm{ind}_{\mathrm{B}_2(\mathbf{Q}_p)}^{\mathrm{GL}_2(\mathbf{Q}_p)} (\chi\mu_{1/\lambda}, \chi\omega^r\mu_\lambda).$$

We can take $R = \overline{\mathbf{F}}_p[\lambda^{\pm 1}, \chi(p)^{\pm 1}]$, and let Σ_R be the set of functions $f \in \Sigma$ with values in R.

In the first case, Σ_R is a free R-module, while in the second case, Σ_R is isomorphic as an R-module to $C^\infty(\mathbf{P}^1(\mathbf{Q}_p), R)$ and hence also free. In either case, if $f \in R$ is nonzero and not a unit and $j \in \mathbf{Z}$, then $f^{j+1} \cdot \Sigma_R$ is a proper $\overline{\mathbf{F}}_p$-linear subrepresentation of $f^j \cdot \Sigma_R$, so that the underlying $\overline{\mathbf{F}}_p$-linear representation of Σ_R is not of finite length. Since $\Sigma_R \subset \Sigma$, the underlying $\overline{\mathbf{F}}_p$-linear representation of Σ is not of finite length, which is a contradiction. This finishes the proof of theorem A.

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