ON TRIANGULABLE TENSOR PRODUCTS OF B-PAIRS AND TRIANGULINE REPRESENTATIONS

by

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Abstract. — We show that if V and V' are two p-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ whose tensor product is trianguline, then V and V' are both potentially trianguline.

Contents

1. Introduction

The notion of a trianguline representation of $G_{\mathbf{Q}_p} = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ was introduced by Colmez [**Col08**] in the context of his work on the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$. Examples of trianguline representations include the semi-stable representations of $G_{\mathbf{Q}_p}$ as well as the p-adic representations of $G_{\mathbf{Q}_p}$ attached to overconvergent cuspidal eigenforms of finite slope (theorem 6.3 of [**Kis03**] and proposition 4.3 of [**Col08**]). The category of all trianguline representations of $G_{\mathbf{Q}_p}$ is stable under extensions, tensor products, and duals. We refer the reader to the book [**BC09**] and the survey [**Ber11**] for a detailed discussion of trianguline representations. Let us at least mention the following analogue of the Fontaine-Mazur conjecture: if V is an irreducible 2-dimensional

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p-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ that is unramified at ℓ for almost all $\ell \neq p$, and whose restriction to a decomposition group at p is trianguline, then V is a twist of the Galois representation attached to an overconvergent cuspidal eigenform of finite slope. This conjecture is a theorem of Emerton (§1.2.2 of [**Eme11**]) under additional technical hypothesis on V. The trianguline property is in general a condition at p reflecting (conjecturally at least) the fact that a p-adic representation comes from a p-adic automorphic form. This theme is pursued, for example, in [**Han17**], [**Ber17**] and [**Con21**].

If K is a finite extension of $\mathbf{Q}_p$, we also have the notion of a trianguline representation of $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$. We say that a representation V of G_K is potentially trianguline if there exists a finite extension L/K such that the restriction of V to G_L is trianguline. The goal of this article is to prove the following theorem.

Theorem A. — *If V and V' are two non-zero p-adic representations of $G_{\mathbf{Q}_p}$ whose tensor product is trianguline, then V and V' are both potentially trianguline.*

We now give more details about the contents of this article. The definition of "trianguline" can be given either in terms of (φ, Γ)-modules over the Robba ring, or in terms of B-pairs. In this article, we use the theory of B-pairs, which was introduced in [**Ber08**]. We remark in passing that B-pairs are the same as G_K-equivariant bundles on the Fargues-Fontaine curve [**FF18**]. Let K be a finite extension of $\mathbf{Q}_p$. Let $\mathbf{B}_{\mathrm{dR}}^+$, $\mathbf{B}_{\mathrm{dR}}$ and $\mathbf{B}_{\mathrm{e}} = (\mathbf{B}_{\mathrm{cris}})^{\varphi=1}$ be some of Fontaine's rings of p-adic periods [**Fon94**]. A B-pair is a pair $W = (W_e, W_{\mathrm{dR}}^+)$ where W_e is a free $\mathbf{B}_{\mathrm{e}}$-module of finite rank endowed with a continuous semi-linear action of G_K, and W_{dR}^+ is a G_K-stable $\mathbf{B}_{\mathrm{dR}}^+$-lattice in $W_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{e}}} W_e$. If V is a p-adic representation of G_K, then $W(V) = (\mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} V, \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V)$ is a B-pair. If E is a finite extension of $\mathbf{Q}_p$, the definition of B-pairs can be extended to E-linear objects, and we get objects called $\mathbf{B}_{|K}^{\otimes E}$-pairs in [**BC10**] or E-B-pairs of G_K in [**Nak09**]. They are pairs $W = (W_e, W_{\mathrm{dR}}^+)$ where W_e is a free $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{e}}$-module of finite rank endowed with a continuous semi-linear action of G_K, and W_{dR}^+ is a G_K-stable $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-lattice in $W_{\mathrm{dR}} = (E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}) \otimes_{E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{e}}} W_e$. Note that the action of G_K is E-linear.

We say (definition 1.15 of [**Nak09**]) that a $\mathbf{B}_{|K}^{\otimes E}$-pair W is split triangulable if W is a successive extension of objects of rank 1, triangulable if there exists a finite extension F/E such that the $\mathbf{B}_{|K}^{\otimes F}$-pair $F \otimes_E W$ is split triangulable, and potentially triangulable if there exists a finite extension L/K such that the $\mathbf{B}_{|L}^{\otimes E}$-pair $W|_{G_L}$ is triangulable. If V is a p-adic representation of G_K, we say that V is trianguline if $W(V)$ is triangulable.

Let Δ be a set of rank 1 semi-linear $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{e}}$-representations of G_K. We say that a $\mathbf{B}_{|K}^{\otimes E}$-pair is split Δ-triangulable if it is split triangulable, and the rank 1 $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{e}}$-representations of G_K that come from the triangulation are all in Δ. Let $\Delta(\mathbf{Q}_p)$ be the set of rank 1 $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{e}}$-representations of G_K that extend to $G_{\mathbf{Q}_p}$. Theorem A then results from the following more general result (theorem 6.4), applied to $K = \mathbf{Q}_p$.

Theorem B. — *If X and Y are two non-zero $\mathbf{B}_{|K}^{\otimes E}$-pairs whose tensor product is $\Delta(\mathbf{Q}_p)$-triangulable, then X and Y are both potentially triangulable.*

The proof of theorem B relies on the study of $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{e}}$-representations of G_K as well as on the study of the slopes, weights and cohomology of $\mathbf{B}_{|K}^{\otimes E}$-pairs. The ring $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{e}}$ has many non-trivial units, which makes the study of $\mathbf{B}_{|K}^{\otimes E}$-pairs more difficult than when $E = \mathbf{Q}_p$. Note finally that some of the results of this article already appear in [**DM13**].

2. Reminders and complements

If K is a finite extension of $\mathbf{Q}_p$, let $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$. Let E be a finite Galois extension of $\mathbf{Q}_p$ such that $K \subset E$, and let $\Sigma = \mathrm{Gal}(E/\mathbf{Q}_p)$. Let E_0 be the maximal unramified extension of $\mathbf{Q}_p$ inside E. Let $\mathbf{B}_{\mathrm{dR}}^+$, $\mathbf{B}_{\mathrm{dR}}$, $\mathbf{B}_{\mathrm{cris}}^+$ and $\mathbf{B}_{\mathrm{cris}}$ be Fontaine's rings of p-adic periods (see for instance [**Fon94**]). They are all equipped with an action of $G_{\mathbf{Q}_p}$, and $\mathbf{B}_{\mathrm{cris}}^+$ and $\mathbf{B}_{\mathrm{cris}}$ have in addition a Frobenius map φ. Let $\mathbf{B}_{\mathrm{e}} = (\mathbf{B}_{\mathrm{cris}})^{\varphi=1}$ and $\mathbf{B}_{\mathrm{e},E} = E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{e}}$. The group $G_{\mathbf{Q}_p}$ acts E-linearly on $\mathbf{B}_{\mathrm{e},E}$.

Proposition 2.1. — *The ring $\mathbf{B}_{\mathrm{e},E}$ is a principal ideal domain.*

Proof. — The ring $\mathbf{B}_{\mathrm{e},E}$ is a Bézout domain; for $E = \mathbf{Q}_p$ this is shown in proposition 1.1.9 of [**Ber08**], and the same argument is used to show the general case in lemma 1.6 of [**Nak09**]. By theorem 6.5.2 of [**FF18**], the ring $\mathbf{B}_{\mathrm{e}}$ is a principal ideal domain, and therefore $\mathbf{B}_{\mathrm{e},E}$ is a principal ideal domain as well, since it is a quotient of the polynomial ring $\mathbf{B}_{\mathrm{e}}[X]$, and thus Noetherian. □

Recall that a $\mathbf{B}_{|K}^{\otimes E}$-pair is a pair $W = (W_e, W_{\mathrm{dR}}^+)$ where W_e is a free $\mathbf{B}_{\mathrm{e},E}$-module of finite rank endowed with a continuous semi-linear action of G_K, and W_{dR}^+ is a G_K-stable $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-lattice in $W_{\mathrm{dR}} = (E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}) \otimes_{\mathbf{B}_{\mathrm{e},E}} W_e$.

Proposition 2.2. — *If W_e is a $\mathbf{B}_{\mathrm{e},E}$-representation of G_K, then $(E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}) \otimes_{\mathbf{B}_{\mathrm{e},E}} W_e$ admits an $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-lattice stable under G_K.*

Proof. — See §3.5 of [**Fon04**]. The same argument gives an $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-lattice instead of a $\mathbf{B}_{\mathrm{dR}}^+$-lattice if one starts from an $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}$-representation. □

Recall that Nakamura has classified the $\mathbf{B}_{|K}^{\otimes E}$-pairs of rank 1, under the assumption that E contains the Galois closure of K. Given a character $\delta : K^\times \to E^\times$, he constructs in §1.4 of [**Nak09**] a rank 1 $\mathbf{B}_{|K}^{\otimes E}$-pair $W(\delta)$, that we denote by $\mathbf{B}(\delta)$, and proves that every rank 1 $\mathbf{B}_{|K}^{\otimes E}$-pair is of this form for a unique δ. We have $\mathbf{B}(\delta_1) \otimes \mathbf{B}(\delta_2) = \mathbf{B}(\delta_1\delta_2)$ (§1.4 of [**Nak09**]). We denote by $\mathbf{B}(\delta)_e$ the $\mathbf{B}_{\mathrm{e},E}$-component of $\mathbf{B}(\delta)$.

Recall (see for instance §2 of [**BC10**] or §1.3 of [**Nak09**]) that $\mathbf{B}_{|K}^{\otimes E}$-pairs have slopes. This comes from the equivalence of categories between $\mathbf{B}_{|K}^{\otimes E}$-pairs and (φ, Γ)-modules over the Robba ring, and Kedlaya's constructions and results for φ-modules over the Robba ring (see [**Ked04**]). In particular, one can define the notion of isoclinic (pure of a certain slope) $\mathbf{B}_{|K}^{\otimes E}$-pairs. For example, if V is an E-linear representation of G_K, then $W(V) = (\mathbf{B}_{\mathrm{e},E} \otimes_E V, (E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+) \otimes_E V)$ is pure of slope 0, and every $\mathbf{B}_{|K}^{\otimes E}$-pair that is pure of slope 0 is of this form (proposition 2.2 of [**BC10**]).

We have the following slope filtration theorem (see theorem 2.1 of [**BC10**]).

Theorem 2.3. — *If W is a $\mathbf{B}_{|K}^{\otimes E}$-pair, there is a canonical filtration $\{0\} = W_0 \subset W_1 \subset \cdots \subset W_\ell = W$ by sub $\mathbf{B}_{|K}^{\otimes E}$-pairs such that*

1. *for every $1 \leqslant i \leqslant \ell$, the quotient W_i/W_{i-1} is isoclinic;*
2. *if s_i is the slope of W_i/W_{i-1}, then $s_1 < s_2 < \cdots < s_\ell$.*

The following proposition gathers the results that we need concerning slopes of $\mathbf{B}_{|K}^{\otimes E}$-pairs. Recall that $\mathrm{Hom}(X,Y) = (\mathrm{Hom}_{E\otimes_{\mathbf{Q}_p}\mathbf{B}_\mathrm{e}}(X_e, Y_e), \mathrm{Hom}_{E\otimes_{\mathbf{Q}_p}\mathbf{B}_{\mathrm{dR}}^+}(X_{\mathrm{dR}}^+, Y_{\mathrm{dR}}^+))$.

Proposition 2.4. — *If X is pure of slope s and Y is pure of slope t, then*

1. $\mathrm{Hom}(X,Y)$ *is pure of slope $t-s$ and $X \otimes Y$ is pure of slope $s+t$;*
2. *if X and Y have the same rank and $X \subset Y$ and $s = t$, then $X = Y$;*
3. *if Y is a direct summand of X, then $s = t$.*

Proof. — For (1), see theorem 6.10 and proposition 5.13 of [**Ked04**]. For (2), we can take determinants and assume that X and Y are of rank 1. The claim is then proposition 2.3 of [**Ber08**]. Item (3) follows from the fact that if $X = Y \oplus Z$, then the set of slopes of X is the union of those of Y and Z (proposition 5.13 of [**Ked04**]). □

3. The ring $\mathbf{B}_{\mathrm{e},E}$

Recall that $\mathbf{B}_{\mathrm{e},E} = E \otimes_{\mathbf{Q}_p} \mathbf{B}_\mathrm{e}$. In this section, we determine the units of $\mathbf{B}_{\mathrm{e},E}$ and study the rank 1 $\mathbf{B}_{\mathrm{e},E}$-representations of G_E. Let $q = p^h$ be the cardinality of the residue field of $\mathcal{O}_E$, so that $E_0 = \mathbf{Q}_{p^h}$. Let $\varphi_E : E \otimes_{E_0} \mathbf{B}_{\mathrm{cris}} \to E \otimes_{E_0} \mathbf{B}_{\mathrm{cris}}$ be the map $\mathrm{Id} \otimes \varphi^h$.

Proposition 3.1. — *We have an exact sequence*

$$0 \to E \to \mathbf{B}_{\mathrm{e},E} \to (E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}})/(E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+) \to 0.$$

Proof. — This follows from tensoring by E the usual fundamental exact sequence $0 \to \mathbf{Q}_p \to \mathbf{B}_{\mathrm{e}} \to \mathbf{B}_{\mathrm{dR}}/\mathbf{B}_{\mathrm{dR}}^+ \to 0$ (proposition 1.17 of [**BK90**]). □

Proposition 3.2. — *The natural map $\mathbf{B}_{\mathrm{e},E} \to (E \otimes_{E_0} \mathbf{B}_{\mathrm{cris}})^{\varphi_E=1}$ is an isomorphism.*

Proof. — Since φ_E is E-linear, we have $(E \otimes_{E_0} \mathbf{B}_{\mathrm{cris}})^{\varphi_E=1} = E \otimes_{E_0} \mathbf{B}_{\mathrm{cris}}^{\varphi^h=1}$ and it is therefore enough to prove that $\mathbf{B}_{\mathrm{cris}}^{\varphi^h=1} = \mathbf{Q}_{p^h} \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{cris}}^{\varphi=1}$. The group $\mathrm{Gal}(\mathbf{Q}_{p^h}/\mathbf{Q}_p)$ acts $\mathbf{Q}_{p^h}$-semi-linearly on $\mathbf{B}_{\mathrm{cris}}^{\varphi^h=1}$ via φ, and the claim follows from Galois descent (Speiser's lemma). □

Remark 3.3. — The isomorphism of proposition 3.2 is G_E-equivariant.

In addition, if $g \in G_{\mathbf{Q}_p}$ acts by $\mathrm{Id} \otimes g$ on $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{e}}$, then it acts by $\mathrm{Id} \otimes g\varphi^{-n(g)}$ on $(E \otimes_{E_0} \mathbf{B}_{\mathrm{cris}})^{\varphi_E=1}$ (where $n(g)$ is defined below).

Let π be a uniformizer of $\mathcal{O}_E$, and let χ_π denote the Lubin-Tate character $\chi_\pi : G_E \to \mathcal{O}_E^\times$ attached to π. For each $\tau \in \Sigma = \mathrm{Gal}(E/\mathbf{Q}_p)$, let $n(\tau)$ be the element of $\{0, \dots, h-1\}$ such that $\tau = \varphi^{n(\tau)}$ on E_0. Let $t_\tau \in E \otimes_{E_0} \mathbf{B}_{\mathrm{cris}}^+$ denote the element constructed in §5 of [**Ber16**], where (in the notation of [**Ber16**]) we take $F = E$. We have $t_\tau = (\tau \otimes \varphi^{n(\tau)})(t_{\mathrm{Id}})$. The element t_{Id} is also denoted by t_π in [**Ber16**], and it is the same as the element t_E constructed in §9 of [**Col02**]. The usual t of p-adic Hodge theory is $t = t_{\mathbf{Q}_p}$ for $\pi = p$.

For each $\sigma \in \Sigma$, we have a map $E \otimes_{E_0} \mathbf{B}_{\mathrm{cris}}^+ \to \mathbf{B}_{\mathrm{dR}}^+$ given by $x \mapsto (\sigma \otimes \varphi^{n(\sigma)})(x)$, followed by the natural injection of $E \otimes_{E_0} \mathbf{B}_{\mathrm{cris}}^+$ in $\mathbf{B}_{\mathrm{dR}}^+$ (theorem 4.2.4 of [**Fon94**]). Finally, note that $E \cdot \widehat{\mathbf{Q}}_p^{\mathrm{nr}} = E \otimes_{E_0} \widehat{\mathbf{Q}}_p^{\mathrm{nr}}$ is contained in $E \otimes_{E_0} \mathbf{B}_{\mathrm{cris}}^+$.

Proposition 3.4. — *Let the notation be as above.*

1. *We have $\varphi_E(t_\tau) = \tau(\pi) \cdot t_\tau$ and $g(t_\tau) = \tau(\chi_\pi(g)) \cdot t_\tau$ if $g \in G_E$;*
2. *the t-adic valuation of the σ-component of the image of t_τ via the map*

$$E \otimes_{E_0} \mathbf{B}_{\mathrm{cris}}^+ \to E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}} = \prod_{\sigma \in \Sigma} \mathbf{B}_{\mathrm{dR}}$$

given by $x \mapsto \{(\sigma \otimes \varphi^{n(\sigma)})(x)\}_{\sigma \in \Sigma}$ is 1 if $\sigma = \tau^{-1}$ and 0 otherwise;

3. *there exists $u \in (E \cdot \widehat{\mathbf{Q}}_p^{\mathrm{nr}})^\times$ such that $\prod_{\tau \in \Sigma} t_\tau = u \cdot t$ in $E \otimes_{E_0} \mathbf{B}_{\mathrm{cris}}$.*

Proof. — Since $t_\tau = (\tau \otimes \varphi^{n(\tau)})(t_{\mathrm{Id}})$, it is enough to check (1) for $\tau = \mathrm{Id}$. The corresponding statement is at the end of §3 of [**Ber16**] (page 3578). Likewise, (2) follows from the case $\tau = \mathrm{Id}$. That case now follows from (1) and the fact that the Hodge-Tate weight of χ_π is 1 at $\sigma = \mathrm{Id}$ and 0 at $\sigma \neq \mathrm{Id}$. Finally, we have $\mathrm{N}_{E/\mathbf{Q}_p}(\chi_\pi) = \chi_{\mathrm{cyc}}\eta$ where $\eta : G_E \to \mathbf{Q}_p^\times$ is unramified, and by (1), this implies (3). □

Note that $t_\tau^{-1} \in E \otimes_{E_0} \mathbf{B}_{\mathrm{cris}}$ since t_τ divides t in $\mathbf{B}_{\mathrm{cris}}^+$ by (3) of proposition 3.4.

Proposition 3.5. — *If $\mathbf{n} = \{n_\tau\}_{\tau\in\Sigma}$ is a tuple of integers whose sum is 0, then there exists $u_{\mathbf{n}} \in (E \cdot \widehat{\mathbf{Q}}_p^{\mathrm{nr}})^\times$ such that $u = \prod_{\tau\in\Sigma} t_\tau^{n_\tau} u_{\mathbf{n}}$ belongs to $\mathbf{B}_{\mathrm{e},E}$. The element u is a unit of $\mathbf{B}_{\mathrm{e},E}$ and every unit of $\mathbf{B}_{\mathrm{e},E}$ is of this form up to multiplication by $E^\times$.*

Proof. — Let $w = \varphi_E(\prod_{\tau\in\Sigma} t_\tau^{n_\tau})/\prod_{\tau\in\Sigma} t_\tau^{n_\tau} = \prod_{\tau\in\Sigma} \tau(\pi)^{n_\tau}$ by (1) of proposition 3.4. Since $\sum_{\tau\in\Sigma} n_\tau = 0$, we have $w \in \mathcal{O}_E^\times$. There exists $u_{\mathbf{n}} \in (E \cdot \widehat{\mathbf{Q}}_p^{\mathrm{nr}})^\times$ such that $\varphi_E(u_{\mathbf{n}})/u_{\mathbf{n}} = w^{-1}$, and then $u = \prod_{\tau\in\Sigma} t_\tau^{n_\tau} u_{\mathbf{n}}$ belongs to $\mathbf{B}_{\mathrm{e},E}$. The inverse of u is $\prod_{\tau\in\Sigma} t_\tau^{-n_\tau} u_{\mathbf{n}}^{-1}$ which also belongs to $\mathbf{B}_{\mathrm{e},E}$, so that $u \in \mathbf{B}_{\mathrm{e},E}^\times$.

We now show that every $u \in \mathbf{B}_{\mathrm{e},E}^\times$ is of this form. Let n_τ be the t-adic valuation in $\mathbf{B}_{\mathrm{dR}}$ of the τ^{-1}-component $u_{\tau^{-1}} = (\tau^{-1} \otimes \mathrm{Id})(u)$ of the image of $u \in E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{e}}$ in $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}} = \prod_{\sigma\in\Sigma} \mathbf{B}_{\mathrm{dR}}$. Note that $u_\sigma \in \mathbf{B}_{\mathrm{e},E}^\times$ for all $\sigma \in \Sigma$ and that $\prod_{\sigma\in\Sigma} u_\sigma \in (\mathbf{B}_{\mathrm{e},E}^\times)^\Sigma = \mathbf{B}_{\mathrm{e}}^\times$. We have $\mathbf{B}_{\mathrm{e}}^\times = \mathbf{Q}_p^\times$ by lemma 1.1.8 of [**Ber08**], so that $\sum_{\tau\in\Sigma} n_\tau = 0$. By (2) of proposition 3.4, the element $u \cdot \prod_{\tau\in\Sigma} t_\tau^{-n_\tau} u_{\mathbf{n}}^{-1}$ belongs to $(E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+) \cap \mathbf{B}_{\mathrm{e},E}^\times$, and $(E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+) \cap \mathbf{B}_{\mathrm{e},E}^\times = E^\times$ by proposition 3.1. □

Recall that an E-linear representation is crystalline or de Rham if the underlying $\mathbf{Q}_p$-linear representation is crystalline or de Rham. We say that a character $\delta : G_E \to E^\times$ is $\mathbf{B}_{\mathrm{e},E}$-admissible if there exists $y \in \mathbf{B}_{\mathrm{e},E} \setminus \{0\}$ such that $\delta(g) = g(y)/y$. Such a character is then crystalline, hence also de Rham.

Proposition 3.6. — *If $y \in \mathbf{B}_{\mathrm{e},E} \setminus \{0\}$ is such that $y \cdot \mathbf{B}_{\mathrm{e},E}$ is stable under G_E, then $y \in \mathbf{B}_{\mathrm{e},E}^\times$ and there exists $n_\tau \in \mathbf{Z}$ with $\sum_{\tau\in\Sigma} n_\tau = 0$ and $y_0 \in (E \cdot \widehat{\mathbf{Q}}_p^{\mathrm{nr}})^\times$ such that $y = \prod_{\tau\in\Sigma} t_\tau^{n_\tau} y_0$.*

Proof. — If $y \cdot \mathbf{B}_{\mathrm{e},E}$ is stable under G_E, then $g(y)/y \in \mathbf{B}_{\mathrm{e},E}$ for all $g \in G_E$. Note that if $z \in \mathbf{B}_{\mathrm{dR}}^\times$, then $g(z)/z \in \mathbf{B}_{\mathrm{dR}}^+$. This implies that $g(y)/y \in \mathbf{B}_{\mathrm{e},E} \cap (E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+)$. By proposition 3.1, $g(y)/y \in E^\times$. The map $\delta : G_E \to E^\times$ given by $\delta(g) = g(y)/y$ is a crystalline character of G_E, and hence of the form $\prod_{\tau\in\Sigma} \tau(\chi_\pi)^{n_\tau}\eta_0$ where $n_\tau \in \mathbf{Z}$ and $\eta_0 : G_E \to E^\times$ is unramified. This implies that there exists $y_0 \in (E \cdot \widehat{\mathbf{Q}}_p^{\mathrm{nr}})^\times$ such that $y = \prod_{\tau\in\Sigma} t_\tau^{n_\tau} y_0$. If $y \in \mathbf{B}_{\mathrm{e},E}$, then $\varphi_E(y) = y$ so that $\sum_{\tau\in\Sigma} n_\tau = 0$ by (1) of proposition 3.4, and hence $y \in \mathbf{B}_{\mathrm{e},E}^\times$. □

Corollary 3.7. — *If $\delta : G_E \to E^\times$ is a $\mathbf{B}_{\mathrm{e},E}$-admissible character, then δ is de Rham and the sum of its weights at all $\tau \in \Sigma$ is 0. Conversely, any character $\delta : G_E \to E^\times$*

that is de Rham with the sum of its weights at all $\tau \in \Sigma$ equal to 0 is the product of a $\mathbf{B}_{\mathrm{e},E}$-admissible character by a potentially unramified character.

Proof. — The first assertion follows immediately from proposition 3.6. We now prove the second assertion. If $\delta : G_E \to E^\times$ is de Rham, it is of the form $\prod_{\tau\in\Sigma} \tau(\chi_\pi)^{n_\tau}\eta_0$ where $n_\tau \in \mathbf{Z}$ and $\eta_0 : G_E \to E^\times$ is potentially unramified. Let $\mathbf{n} = \{n_\tau\}_{\tau\in\Sigma}$ and u be the corresponding unit (proposition 3.5). If $g \in G_E$, then $g(u)/u = \prod_{\tau\in\Sigma} \tau(\chi_\pi(g))^{n_\tau}\eta_u(g)$ where $\eta_u : G_E \to E^\times$ is unramified. The second assertion then follows from this. □

A $\mathbf{B}_{\mathrm{e},E}$-representation of G_K is a free $\mathbf{B}_{\mathrm{e},E}$-module of finite rank with a semi-linear and continuous action of G_K (recall that G_K acts linearly on E). If $\delta \in H^1(G_K, \mathbf{B}_{\mathrm{e},E}^\times)$ (for example if $\delta : G_K \to E^\times$ is a character), we denote by $\mathbf{B}_{\mathrm{e},E}(\delta)$ the resulting rank 1 $\mathbf{B}_{\mathrm{e},E}$-representation of G_K.

Proposition 3.8. — *If W_e is a $\mathbf{B}_{\mathrm{e},E}$-representation of G_K, and if X_e is a sub $\mathbf{B}_{\mathrm{e},E}$-module of W_e stable under G_K, then X_e is a free $\mathbf{B}_{\mathrm{e},E}$-module, and it is saturated in W_e.*

Proof. — See lemma 1.10 of [**Nak09**]. □

Proposition 3.9. — *If W is a rank 1 $\mathbf{B}_{\mathrm{e},E}$-representation of G_E, then there exists $\delta : G_E \to E^\times$ such that $W = \mathbf{B}_{\mathrm{e},E}(\delta)$.*

Proof. — If we choose a basis w of W, then $g(w) = \delta(g)w$ with $\delta(g) \in \mathbf{B}_{\mathrm{e},E}^\times$, so that $\delta(g)$ is of the form $\prod_{\tau\in\Sigma} t_\tau^{n_\tau(g)}u_{\mathbf{n}(g)}$ by proposition 3.5. Since $\delta(gh) = \delta(g)g(\delta(h))$, (1) of proposition 3.4 implies that the maps $n_\tau : G_E \to \mathbf{Z}$ are continuous homomorphisms. They are therefore trivial, and this implies that $\delta(g) \in E^\times$. □

Remark 3.10. — The character δ in proposition 3.9 is not unique, since it can be multiplied by any $\mathbf{B}_{\mathrm{e},E}$-admissible character of G_E.

Remark 3.11. — If $K \neq E$, it is not necessarily true that every rank 1 $\mathbf{B}_{\mathrm{e},E}$-representation of G_K is of the form $\mathbf{B}_{\mathrm{e},E}(\delta)$ for a character $\delta : G_K \to E^\times$.

Proof. — Take $E = \mathbf{Q}_p(\sqrt{p})$ and $K = \mathbf{Q}_p$ and $W = (E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{cris}})^{\varphi=\pi} = t_{\mathrm{Id}} \cdot \mathbf{B}_{\mathrm{e},E}$. The E-linear action of $G_{\mathbf{Q}_p}$ on W is given by the map $\delta : g \mapsto g(t_{\mathrm{Id}})/t_{\mathrm{Id}}$. If $g \in G_E$, then $\delta(g) = \chi_\pi(g)$. If $u = t_{\mathrm{Id}}^n t_\tau^{-n} u_{n,-n} \in \mathbf{B}_{\mathrm{e},E}^\times$ as in proposition 3.5, and $g \notin G_E$, then $g(ut_{\mathrm{Id}})/ut_{\mathrm{Id}} = t_{\mathrm{Id}}^{-2n-1}t_\tau^{2n+1}v$ with $v \in (E \cdot \widehat{\mathbf{Q}}_p^{\mathrm{nr}})^\times$. Therefore, there is no character $\eta : G_{\mathbf{Q}_p} \to E^\times$ such that $W = \mathbf{B}_{\mathrm{e},E}(\eta)$.

Note that W is the $\mathbf{B}_{\mathrm{e},E}$-component of the $\mathbf{B}_{|K}^{\otimes E}$-pair W_0^{-1} of §1.4 of [**Nak09**]. □

Remark 3.12. — The results of this section provide a new proof of proposition 2.1.

Proof. — By theorem 6.5.2 of [**FF18**], the ring $(E \otimes_{E_0} \mathbf{B}_{\mathrm{cris}}^+[1/t_{\mathrm{Id}}])^{\varphi_E=1}$ is a PID. Since we have shown $\mathbf{B}_{\mathrm{e},E}$ is a localization of $(E \otimes_{E_0} \mathbf{B}_{\mathrm{cris}}^+[1/t_{\mathrm{Id}}])^{\varphi_E=1}$, it is a PID. □

Proposition 3.13. — *We have* $\mathrm{Frac}(\mathbf{B}_{\mathrm{e},E})^{G_K} = E$.

Proof. — Take $x/y \in \mathrm{Frac}(\mathbf{B}_{\mathrm{e},E})^{G_K}$ with $x, y \in \mathbf{B}_{\mathrm{e},E}$ coprime. If $g \in G_K$, then $g(x)y = xg(y)$ so that x divides $g(x)$ and y divides $g(y)$ in $\mathbf{B}_{\mathrm{e},E}$ (recall that $\mathbf{B}_{\mathrm{e},E}$ is a PID). By proposition 3.6, x and y belong to $\mathbf{B}_{\mathrm{e},E}^\times$. This implies that $x/y \in \mathbf{B}_{\mathrm{e},E}^{G_K} = E$. □

Corollary 3.14. — *If* W_e *is a* $\mathbf{B}_{\mathrm{e},E}$*-representation of* G_K*, then* $\dim_E W_e^{G_K} \leqslant \mathrm{rk}\, W_e$.

Proof. — By a standard argument, proposition 3.13 implies that the map $\mathbf{B}_{\mathrm{e},E} \otimes_E W_e^{G_K} \to W_e$ is injective. This implies the corollary. □

4. Triangulable representations

In this section, we study triangulable $\mathbf{B}_{|K}^{\otimes E}$-pairs and $\mathbf{B}_{\mathrm{e},E}$-representations of G_K. We say that a $\mathbf{B}_{|K}^{\otimes E}$-pair is irreducible if it has no non-trivial saturated sub $\mathbf{B}_{|K}^{\otimes E}$-pair (see §2.1 of [**Ber08**]).

Proposition 4.1. — *If* $W = (W_e, W_{\mathrm{dR}}^+)$ *is an irreducible* $\mathbf{B}_{|K}^{\otimes E}$*-pair, then* W_e *is an irreducible* $\mathbf{B}_{\mathrm{e},E}$*-representation of* G_K.

Proof. — Let X_e be a sub-object of W_e. By proposition 3.8, it is a saturated and free submodule of W_e. The space $X_{\mathrm{dR}}^+ = X_{\mathrm{dR}} \cap W_{\mathrm{dR}}^+$ is an $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$ lattice of X_{dR} stable under G_K. Hence $X = (X_e, X_{\mathrm{dR}}^+)$ is a saturated sub $\mathbf{B}_{|K}^{\otimes E}$-pair of W. □

Corollary 4.2. — *If* W *is a* $\mathbf{B}_{|K}^{\otimes E}$*-pair, then* W *is split triangulable as a* $\mathbf{B}_{|K}^{\otimes E}$*-pair if and only if* W_e *is split triangulable as a* $\mathbf{B}_{\mathrm{e},E}$*-representation of* G_K.

Proof. — It is clear that if W is split triangulable, then so is W_e. Conversely, the proof of proposition 4.1 shows how to construct a triangulation of W from a triangulation of W_e. □

Let Δ be a set of rank 1 semi-linear $\mathbf{B}_{\mathrm{e},E}$-representations of G_K. Recall that a $\mathbf{B}_{|K}^{\otimes E}$-pair is split Δ-triangulable if it is split triangulable, and the rank 1 $\mathbf{B}_{\mathrm{e},E}$-representations of G_K that come from the triangulation are all in Δ.

Proposition 4.3. — *If* $0 \to W' \to W \to W'' \to 0$ *is an exact sequence of* $\mathbf{B}_{|K}^{\otimes E}$*-pairs, then* W *is split* Δ*-triangulable if and only if* W' *and* W'' *are split* Δ*-triangulable.*

Proof. — If W' and W'' are split Δ-triangulable, then W is obviously split Δ-triangulable. We now prove the converse. If W_e admits a triangulation, then so do W'_e and W''_e. By corollary 4.2, W' and W'' are therefore split triangulable. Proposition 3.8 implies that two different triangulations of W_e give rise to two composition series of W_e (seen as a $\mathbf{B}_{\mathrm{e},E}$-representation of G_K). The set of rank 1 $\mathbf{B}_{\mathrm{e},E}$-representations attached to any triangulation of W_e is therefore well-defined up to permutation by the Jordan-Hölder theorem. Hence if W is split Δ-triangulable, then so are W' and W''. □

Proposition 4.4. — *If W_e is an irreducible $\mathbf{B}_{\mathrm{e},E}$-representation of G_K, and $\delta \in H^1(G_K, \mathbf{B}_{\mathrm{e},E}^\times)$, then every surjective map $\pi : \mathrm{End}(W_e) \to \mathbf{B}_{\mathrm{e},E}(\delta)$ of $\mathbf{B}_{\mathrm{e},E}$-representations of G_K is split.*

Proof. — Write $\mathbf{B}_{\mathrm{e},E}(\delta) = \mathbf{B}_{\mathrm{e},E} \cdot e_\delta$, where $g(e_\delta) = \delta(g)e_\delta$ with $\delta(g) \in \mathbf{B}_{\mathrm{e},E}^\times$. Recall that if A is a ring and M is a free A-module, then $\mathrm{End}_A(M)$ is its own dual, for the pairing $(f, g) \mapsto \mathrm{Tr}(fg)$. The map π is therefore of the form $f \mapsto \mathrm{Tr}(fh) \cdot e_\delta$ for some $h \in \mathrm{End}(W_e)$. The map h satisfies $g(h) = \delta(g)^{-1}h$, and therefore gives rise to a G_K-equivariant map $h : W_e \to W_e(\delta)$. Since W_e is irreducible, h is invertible. We can then write $\mathrm{End}(W_e) = \ker(\pi) \oplus \mathbf{B}_{\mathrm{e},E} \cdot h^{-1}$, which shows that π is split. □

Theorem 4.5. — *If W_e is an irreducible $\mathbf{B}_{\mathrm{e},E}$-representation of G_K such that $\mathrm{End}(W_e)$ is split triangulable, then the triangulation of $\mathrm{End}(W_e)$ splits.*

Proof. — Write $\{0\} = X_0 \subset X_1 \subset \cdots \subset X_d = \mathrm{End}(W_e)$, and $X_i/X_{i-1} = \mathbf{B}_{\mathrm{e},E}(\delta_i)$ for some $\delta_i \in H^1(G_K, \mathbf{B}_{\mathrm{e},E}^\times)$. By proposition 4.4, the exact sequence $0 \to X_{d-1} \to \mathrm{End}(W_e) \to \mathbf{B}_{\mathrm{e},E}(\delta_d) \to 0$ is split, and therefore $\mathrm{End}(W_e) = X_{d-1} \oplus \mathbf{B}_{\mathrm{e},E}(\delta_d)$.

Suppose that we have an isomorphism $\mathrm{End}(W_e) = X_j \oplus \mathbf{B}_{\mathrm{e},E}(\delta_{j+1}) \oplus \cdots \oplus \mathbf{B}_{\mathrm{e},E}(\delta_d)$. Let π_j denote the composition $\mathrm{End}(W_e) \to X_j \to \mathbf{B}_{\mathrm{e},E}(\delta_j)$. By proposition 4.4, $\mathrm{End}(W_e) = \ker(\pi_j) \oplus \mathbf{B}_{\mathrm{e},E}(\delta_j)$. We have $\ker(\pi_j) = X_{j-1} \oplus \mathbf{B}_{\mathrm{e},E}(\delta_{j+1}) \oplus \cdots \oplus \mathbf{B}_{\mathrm{e},E}(\delta_d)$, so that $\mathrm{End}(W_e) = X_{j-1} \oplus \mathbf{B}_{\mathrm{e},E}(\delta_j) \oplus \cdots \oplus \mathbf{B}_{\mathrm{e},E}(\delta_d)$. The claim follows by induction. □

Remark 4.6. — Theorem 4.5 is reminiscent of the following result of Chevalley: if G is any group and if X and Y are finite dimensional semi-simple characteristic 0 representations of G, then $X \otimes Y$ is also semi-simple. The same holds for semi-linear representations and, more generally, in any Tannakian category over a field of characteristic 0 [**Del16**].

5. Cohomology of B-pairs

The cohomology of $\mathbf{B}_{|K}^{\otimes E}$-pairs is defined and studied in §2.1 of [**Nak09**]. We recall what we need. Let W be a $\mathbf{B}_{|K}^{\otimes E}$-pair. Nakamura constructs an E-vector space $H^1(G_K, W)$ that has the following properties

1. $H^1(G_K, W) = \mathrm{Ext}^1(\mathbf{B}, W)$ (i.e. it classifies the extensions of $\mathbf{B}_{|K}^{\otimes E}$-pairs);
2. there is an exact sequence of E-vector spaces

$$W_{\mathrm{dR}}^{G_K} \to H^1(G_K, W) \to H^1(G_K, W_e) \oplus H^1(G_K, W_{\mathrm{dR}}^+).$$

If W is a rank 1 $\mathbf{B}_{|K}^{\otimes E}$-pair with $W_e \in \Delta(\mathbf{Q}_p)$, then $W_{\mathrm{dR}}^{G_{\mathbf{Q}_p}}$ is an E-vector space of dimension 1 or 0, depending on whether W_e (extended to $G_{\mathbf{Q}_p}$) is de Rham or not. Since $W_{\mathrm{dR}}^{G_K} = K \otimes_{\mathbf{Q}_p} W_{\mathrm{dR}}^{G_{\mathbf{Q}_p}}$, this implies that $W_{\mathrm{dR}}^{G_K} = \{0\}$ if W is not de Rham. Note that if W is a rank 1 $\mathbf{B}_{|K}^{\otimes E}$-pair with $K \neq \mathbf{Q}_p$, then W may be "partially de Rham" in the sense of [**Din17**], so that in general $W_{\mathrm{dR}}^{G_K}$ can be non-zero even if W is not de Rham.

Proposition 5.1. — *If W_{dR}^+ is a free $E \otimes_{\mathbf{Q}_p} \mathbf{B}_{\mathrm{dR}}^+$-representation of G_K of rank 1, the map $H^1(G_K, W_{\mathrm{dR}}^+) \to H^1(G_K, W_{\mathrm{dR}})$ is injective.*

Proof. — Since we have an exact sequence

$$W_{\mathrm{dR}}^{G_K} \to (W_{\mathrm{dR}}/W_{\mathrm{dR}}^+)^{G_K} \to H^1(G_K, W_{\mathrm{dR}}^+) \to H^1(G_K, W_{\mathrm{dR}}),$$

it is enough to show that $W_{\mathrm{dR}}^{G_K} \to (W_{\mathrm{dR}}/W_{\mathrm{dR}}^+)^{G_K}$ is surjective. To do this, we can replace K by a finite extension L, and in particular we can assume that L contains E. In this case, $W_{\mathrm{dR}}^+|_{G_L}$ is a direct sum of rank 1 $\mathbf{B}_{\mathrm{dR}}^+$-representations of G_L.

Let X_{dR}^+ be a rank 1 $\mathbf{B}_{\mathrm{dR}}^+$-representation of G_L. The L-vector space $X_{\mathrm{dR}}^{G_L}$ is of dimension 0 or 1. If $\dim_L X_{\mathrm{dR}}^{G_L} = 1$, then X_{dR} is de Rham, and the map $X_{\mathrm{dR}}^{G_L} \to (X_{\mathrm{dR}}/X_{\mathrm{dR}}^+)^{G_L}$ is surjective by the same argument as in lemma 3.8.1 of [**BK90**] (see lemma 2.6 of [**Nak09**]). If $\dim_L X_{\mathrm{dR}}^{G_L} = 0$, then for every $i \in \mathbf{Z}$, we have $(t^i X_{\mathrm{dR}}^+/t^{i+1} X_{\mathrm{dR}}^+)^{G_L} = 0$ by proposition 3.21 of [**Fon04**]. This implies that $(X_{\mathrm{dR}}/X_{\mathrm{dR}}^+)^{G_L} = 0$, so that the map $X_{\mathrm{dR}}^{G_L} \to (X_{\mathrm{dR}}/X_{\mathrm{dR}}^+)^{G_L}$ is also surjective. □

Corollary 5.2. — *If X is a direct sum of rank 1 $\mathbf{B}_{|K}^{\otimes E}$-pairs, the map $H^1(G_K, X_{\mathrm{dR}}^+) \to H^1(G_K, X_{\mathrm{dR}})$ is injective.*

Recall that every rank 1 $\mathbf{B}_{|K}^{\otimes E}$-pair is of the form $\mathbf{B}(\delta)$ for a unique $\delta : K^\times \to E^\times$.

Proposition 5.3. — *If a $\mathbf{B}_{|K}^{\otimes E}$-pair W is split $\Delta(\mathbf{Q}_p)$-triangulable, with subquotients $\{\mathbf{B}(\delta_i)\}_i$ such that $\mathbf{B}(\delta_i\delta_j^{-1})$ is not de Rham for any $i \neq j$, and if the corresponding* *triangulation of* W_e *splits as a direct sum of* 1*-dimensional* $\mathbf{B}_{\mathrm{e},E}$*-representations, then the triangulation of* W *splits.*

Proof. — Let $0 = W_0 \subset W_1 \subset \cdots \subset W_d = W$ be the given triangulation of W. We prove by induction on j that $W_j = \mathbf{B}(\delta_1) \oplus \cdots \oplus \mathbf{B}(\delta_j)$. This is true for $j = 1$, assume it holds for $j-1$. Write $0 \to W_{j-1} \to W_j \to \mathbf{B}(\delta_j) \to 0$ and $W_{j-1} = \mathbf{B}(\delta_1) \oplus \cdots \oplus \mathbf{B}(\delta_{j-1})$. Let $X = W_{j-1}(\delta_j^{-1})$ and $Y = W_j(\delta_j^{-1})$. The $\mathbf{B}_{|K}^{\otimes E}$-pair Y corresponds to a class in $H^1(G_K, X)$. The $\mathbf{B}_{\mathrm{e},E}$-representation Y_e is split, and therefore so is Y_{dR}. By corollary 5.2, so is Y_{dR}^+. The class of Y in $H^1(G_K, X)$ is therefore in the kernel of $H^1(G_K, X) \to H^1(G_K, X_e) \oplus H^1(G_K, X_{\mathrm{dR}}^+)$. Since $X_{\mathrm{dR}}^{G_K} = 0$ by hypothesis, Nakamura's exact sequence (2) above implies that the class of Y is trivial and hence $W_j = W_{j-1} \oplus \mathbf{B}(\delta_j)$. The proposition follows by induction. □

6. Proof of the main theorem

In this section, we prove theorem B. Let F be a finite extension of E of degree $\geqslant 2$, and write $F \otimes_E F = \oplus_i F_i$. There are at least two summands since F itself is one of them.

Proposition 6.1. — *Let* F/E *be as above, and let* W *be an* F*-linear representation of* G_K. *We have* $F \otimes_E W = \oplus_i (F_i \otimes_F W)$ *as* F*-linear representations of* G_K.

Proof. — We have $F \otimes_E W = (F \otimes_E F) \otimes_F W = \oplus_i (F_i \otimes_F W)$. □

Corollary 6.2. — *If* W *is a* $\mathbf{B}_{\mathrm{e},E}$*-representation of* G_K *that has an* F*-linear structure, then* W *becomes reducible after extending scalars from* E *to* F.

Let us say that a $\mathbf{B}_{|K}^{\otimes E}$-pair W is completely irreducible if $(F \otimes_E W)|_{G_L}$ is an irreducible $\mathbf{B}_{|L}^{\otimes F}$-pair for all finite extensions F of E and L of K.

Proposition 6.3. — *If* $K = E$ *and if* X *and* Y *are two completely irreducible* $\mathbf{B}_{|K}^{\otimes E}$*-pairs such that* $\mathrm{Hom}(X, Y)$ *is split* $\Delta(\mathbf{Q}_p)$*-triangulable, then* X *and* Y *are of rank* 1.

Proof. — Let $\{\mathbf{B}(\delta_i)\}_i$ be the rank 1 subquotients of the triangulation of $\mathrm{Hom}(X, Y)$. We have an inclusion $\mathbf{B}(\delta_1) \subset \mathrm{Hom}(X, Y)$. This gives rise to a non-zero map $X \to Y(\delta_1^{-1})$ of $\mathbf{B}_{|K}^{\otimes E}$-pairs. Write $\mathbf{B}(\delta_1)_e = \mathbf{B}_{\mathrm{e},E}(\mu_1)$ for some $\mu_1 : G_K \to \mathbf{B}_{\mathrm{e},E}^\times$ (recall that $K = E$). Since X and Y are irreducible, X_e and Y_e are irreducible $\mathbf{B}_{\mathrm{e},E}$-representations of G_K (proposition 4.1), and the map $X_e \to Y_e(\mu_1^{-1})$ is therefore an isomorphism. This implies that $\mathrm{Hom}(X_e, Y_e) = \mathrm{End}(X_e)(\mu_1)$, so that $\mathrm{End}(X_e)$ is split triangulable. By theorem 4.5, the triangulation of $\mathrm{End}(X_e)$ splits. The triangulation of $\mathrm{Hom}(X_e, Y_e) = \mathrm{End}(X_e)(\mu_1)$ therefore also splits. Let n be the common rank of X and Y.

Suppose that none of the $\mathbf{B}(\delta_i\delta_j^{-1})$ are de Rham for any $i \neq j$. By proposition 5.3 applied to $W = \mathrm{Hom}(X, Y)$, the triangulation of $\mathrm{Hom}(X, Y)$ splits. We can therefore write $\mathrm{Hom}(X, Y) = \oplus_i \mathbf{B}(\delta_i)$. Since X and Y are both irreducible, they are pure of some slopes s and t by theorem 2.3. The $\mathbf{B}_{|K}^{\otimes E}$-pair $\mathrm{Hom}(X, Y)$ is then pure of slope $t - s$ by (1) of proposition 2.4. By (3) of ibid, each of the $\mathbf{B}(\delta_i)$ is also pure of slope $t - s$. Each $\mathbf{B}(\delta_i)$ gives rise to a map $X \to Y(\delta_i^{-1})$, which is an isomorphism of $\mathbf{B}_{|K}^{\otimes E}$-pairs by (2) of ibid, since X and $Y(\delta_i^{-1})$ are both pure of slope s. By taking determinants, we get $\delta_i^n = \det(Y)\det(X)^{-1}$ for every i. This implies that $(\delta_i\delta_j^{-1})^n = 1$ so that $\delta_i\delta_j^{-1}$ is of finite order, and $\mathbf{B}(\delta_i\delta_j^{-1})$ is de Rham (lemma 4.1 of [**Nak09**]), contradicting our assumption.

Therefore, one of the $\mathbf{B}(\delta_i\delta_j^{-1})$ is de Rham for some $i \neq j$. Write $\mathbf{B}(\delta_k)_e = \mathbf{B}_{\mathrm{e},E}(\mu_k)$ where the μ_k are characters $G_K \to E^\times$ (recall that $K = E$), so that $\mathrm{End}(X_e)(\mu_1) = \oplus_k \mathbf{B}_{\mathrm{e},E}(\mu_k)$ as $\mathbf{B}_{\mathrm{e},E}$-representations of G_K. The fact that $\mathbf{B}(\delta_i\delta_j^{-1})$ is de Rham implies that $\mu_i\mu_j^{-1}$ is de Rham. We then have $X_e = X_e(\mu_1\mu_i^{-1}) = X_e(\mu_1\mu_j^{-1})$, so that $X_e = X_e(\mu_i\mu_j^{-1})$. By taking determinants, we find that $\mathbf{B}_{e,E}((\mu_i\mu_j^{-1})^n) = \mathbf{B}_{e,E}$ and therefore by corollary 3.7, $(\mu_i\mu_j^{-1})^n : G_K \to E^\times$ is de Rham and the sum of its weights is 0. This implies that the sum of the weights of $\mu_i\mu_j^{-1} : G_K \to E^\times$ is 0. By corollary 3.7, $\mu_i\mu_j^{-1} = \chi\eta$ with $\chi : G_K \to E^\times$ a $\mathbf{B}_{e,E}$-admissible character and $\eta : G_K \to E^\times$ potentially unramified. Since $X_e(\chi\eta) = X_e$ and $X_e(\chi) = X_e$, we get $X_e(\eta) = X_e$. By taking determinants, we get that η^n is $\mathbf{B}_{e,E}$-admissible. Since η^n is also potentially unramified, and $\mathbf{B}_{\mathrm{e},E} \cap (\overline{\mathbf{Q}}_p \cdot \widehat{\mathbf{Q}}_p^{\mathrm{nr}}) = E$, it is trivial. Hence η is a character of finite order of G_K, and so there exists a finite extension L of K such that $\mu_i = \chi\mu_j$ on G_L.

The space $\mathrm{End}(X_e)(\mu_1)$ contains $\mathbf{B}_{\mathrm{e},E}(\mu_j) \oplus \mathbf{B}_{\mathrm{e},E}(\mu_i)$, which is isomorphic to $\mathbf{B}_{\mathrm{e},E}(\mu_j) \oplus \mathbf{B}_{\mathrm{e},E}(\mu_j)$ after restricting to G_L. Let f and g be the two resulting isomorphisms $X_e \to X_e(\mu_1\mu_j^{-1})$. The map $h = f^{-1} \circ g : X_e \to X_e$ is G_L-equivariant and is not in $E^\times \cdot \mathrm{Id}$ since f and g are $\mathbf{B}_{\mathrm{e},E}$-linearly independent. Therefore, $\mathrm{End}(X_e)^{G_L}$ is strictly larger than E.

Since $X_e|_{G_L}$ is irreducible, Schur's lemma and corollary 3.14 imply that $\mathrm{End}(X_e)^{G_L}$ contains a field F such that $[F : E] \geqslant 2$ (for example, $F = E[h]$). Hence $X_e|_{G_L}$ has an F-linear structure. Corollary 6.2 implies that $(F \otimes_E X_e)|_{G_L}$ is reducible. By proposition 4.1, X is not completely irreducible. This is a contradiction, so X had to be of rank 1. Since X and Y have the same rank, we are done. □

We now recall and prove theorem B. A strict sub-quotient of a $\mathbf{B}_{|K}^{\otimes E}$-pair is a quotient of a saturated sub $\mathbf{B}_{|K}^{\otimes E}$-pair.

Theorem 6.4. — *If X and Y are two non-zero $\mathbf{B}_{|K}^{\otimes E}$-pairs whose tensor product is $\Delta(\mathbf{Q}_p)$-triangulable, then X and Y are both potentially triangulable.*

Proof. — We can replace E and K by finite extensions F and L if necessary, and write X and Y as successive extensions of completely irreducible $\mathbf{B}_{|L}^{\otimes F}$-pairs with $F = L$. If X' and Y' are two strict sub-quotients of X and Y, then $X' \otimes Y'$ is a strict sub-quotient of $X \otimes Y$, and it is $\Delta(\mathbf{Q}_p)$-triangulable by proposition 4.3. Proposition 6.3, applied to $(X')^*$ and Y' so that $X' \otimes Y' = \mathrm{Hom}((X')^*, Y')$, tells us that X' and Y' are of rank 1.

Hence the $\mathbf{B}_{|L}^{\otimes F}$-pairs $(F \otimes_E X)|_{G_L}$ and $(F \otimes_E Y)|_{G_L}$ are split triangulable. □

Corollary 6.5. — *If X_e and Y_e are two $\mathbf{B}_{\mathrm{e},E}$-representations of G_K whose tensor product is triangulable, with the rank 1 sub-quotients extending to $\mathbf{B}_{\mathrm{e},E}$-representations of $G_{\mathbf{Q}_p}$, then X_e and Y_e are both potentially triangulable.*

Proof. — By proposition 2.2, X_e and Y_e extend to $\mathbf{B}_{|K}^{\otimes E}$-pairs. The result follows from corollary 4.2 and theorem 6.4. □

We finish with an example of a representation V such that $V \otimes_E V$ is trianguline, but V itself is not trianguline. This shows that the "potentially" in the statement of theorem A cannot be avoided. Let Q_8 denote the quaternion group. If $p \equiv 3 \bmod 4$, there is a Galois extension $K/\mathbf{Q}_p$ such that $\mathrm{Gal}(K/\mathbf{Q}_p) = Q_8$ (see II.3.6 of [**JY88**]). Choose such a p and K, and let E be a finite extension of $\mathbf{Q}_p$ containing $\sqrt{-1}$. The group Q_8 has a (unique) irreducible 2-dimensional E-linear representation, which we inflate to a representation V of $G_{\mathbf{Q}_p}$. One can check that $V \otimes_E V$ is a direct sum of characters, hence trianguline, and that the semi-linear representation $\mathrm{Frac}(\mathbf{B}_{\mathrm{e},E}) \otimes_E V$ is irreducible. This holds for all E as above, so that V is not trianguline.

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