Let $V$ be a $\mathbb{Q}_p$-linear representation of $G_K$. In this appendix we prove the following theorem.

**Theorem A.1.** — If $V$ is semistable and all its Hodge-Tate weights are $\geq 2$, then $H^2(G_K, V) = 0$.

Let $D(V)$ be Fontaine’s $(\varphi, \Gamma)$-module attached to $V$ [Fon90]. It comes with a Frobenius map $\varphi$ and an action of $\Gamma_K$. Let $H_K = \text{Gal}(\overline{K}/K(\mu_{p^\infty}))$ and let $I_K = \text{Gal}(\overline{K}/K^{\text{nr}})$. The injectivity of the restriction map $H^2(G_K, V) \to H^2(G_L, V)$ for $L/K$ finite allows us to replace $K$ by a finite extension, so that we can assume that $H_K I_K = G_K$ and that $\Gamma_K \simeq \mathbb{Z}_p$. Let $\gamma$ be a topological generator of $\Gamma_K$. Recall ($\S$1.5 of [CC99]) that we have a map $\psi : D(V) \to D(V)$.

Ideally, our proof of this theorem would go as follows. We use the Hochschild-Serre spectral sequence

$$H^i(G_K/I_K, H^j(I_K, V|_{I_K})) \Rightarrow H^{i+j}(G_K, V)$$

and, interpreting Galois cohomology in terms of $(\varphi, \Gamma)$-modules, we compute that $H^2(I_K, V|_{I_K}) = 0$ and $H^1(I_K, V|_{I_K}) = \hat{K}^{\text{nr}} \otimes_K D_{\text{dR}}(V)$. We conclude since, by Hilbert 90, $H^1(G_K/I_K, H^1(I_K, V|_{I_K})) = 0$. However, we do not, in general, have Hochschild-Serre spectral sequences for continuous cohomology. We mimic thus the above argument with direct computations on continuous cocycles (again using $(\varphi, \Gamma)$-modules). Laurent Berger is grateful to Kevin Buzzard for discussions related to the above spectral sequence.

**Lemma A.2.** — 1. If $V$ is a representation of $G_K$, then there is an exact sequence

$$0 \to D(V)^{(\psi - 1)/\Gamma_K} \to H^1(G_K, V) \to (D(V)/(\psi - 1))^{\Gamma_K} \to 0;$$
2. We have $H^2(G_K, V) = D(V)/(\psi - 1, \gamma - 1)$.

Proof. — See I.5.5 and II.3.2 of [CC99].

**Lemma A.3.** — We have $D(V|_{I_K})/(\psi - 1) = 0$

Proof. — Since $V|_{I_K}$ corresponds to the case when $k$ is algebraically closed, see the proof of Lemma VI.7 of [Ber01].

Let $\gamma_I$ denote a generator of $\Gamma_{\hat{K}^nr}$.

**Lemma A.4.** — The natural map $D(V|_{I_K})^{\psi=1}/(\gamma_I - 1) \rightarrow (D(V|_{I_K})/(\gamma_I - 1))^{\psi=1}$ is an isomorphism if $V^{I_K} = 0$.

Proof. — This map is part of the six term exact sequence that comes from the map $\gamma_I - 1$ applied to $0 \rightarrow D(V|_{I_K})^{\psi=1} \rightarrow D(V|_{I_K}) \xrightarrow{\psi - 1} D(V|_{I_K}) \rightarrow 0$. Its kernel is included in $D(V|_{I_K})^{\gamma_I=1}$ which is 0, since $V^{I_K} = 0$ (note that the inclusion $(\hat{K}^nr \otimes V)^{G_K} \subseteq (\hat{E}^{nr} \otimes V)^{G_K} = D(V)^{G_K}$ is an isomorphism).

Suppose that $x \in D(V)/(\psi - 1, \gamma - 1)$. If $\tilde{x} \in D(V)$ lifts $x$, then Lemma A.3 gives us an element $y \in D(V|_{I_K})$ such that $(\psi - 1)y = \tilde{x}$. Define a cocycle $\delta(x) \in Z^1(G_K/I_K, D(V|_{I_K})^{\psi=1}/(\gamma_I - 1))$ by $\delta(x) : \bar{g} \mapsto (g - 1)(y)$ if $g \in G_K$ lifts $\bar{g} \in G_K/I_K$.

**Proposition A.5.** — If $V^{I_K} = 0$, then the map

$$\delta : D(V)/(\psi - 1, \gamma - 1) \rightarrow H^1(G_K/I_K, (D(V|_{I_K})/(\gamma_I - 1))^{\psi=1})$$

is well-defined and injective.

Proof. — We first check that $\delta(x)(g) \in (D(V|_{I_K})/(\gamma_I - 1))^{\psi=1}$. We have $(\psi - 1)(g - 1)(y) = (g - 1)(x)$. If we write $g = ih \in I_K H_K$, then $g - 1 = ih - 1 = (i - 1)x \in (\gamma_I - 1)D(V|_{I_K})$ since $\gamma_I - 1$ divides the image of $i - 1$ in $Z_p[\Gamma_{\hat{K}^nr}]$. This implies that $\delta(x)(g) \in (D(V|_{I_K})/(\gamma_I - 1))^{\psi=1}$.

We now check that $\delta(x)$ does not depend on the choices. If we choose another lift $g' \in G_K$ of $\bar{g} \in G_K/I_K$, then $g' = ig$ for some $i \in I_K$ and $(g' - 1)y - (g - 1)y = (i - 1)gy \in (\gamma_I - 1)D(V|_{I_K})$ since $\gamma_I - 1$ divides the image of $i - 1$ in $Z_p[\Gamma_{\hat{K}^nr}]$. If we choose another $g'$ such that $(\psi - 1)y' = \tilde{x}$, then $y - y' \in D(V|_{I_K})^{\psi=1}$ so that $\delta$ and $\delta'$ are cohomologous. Finally, if $\tilde{x}'$ is another lift of $x$, then $\tilde{x}' - \tilde{x} = (\gamma_I - 1)a + (\psi - 1)b$ with $a, b \in D(V)$. We can then take $y' = y + b + (\gamma_G - 1)c$ where $(\psi - 1)c = a$. We then have $(g - 1)y' = (g - 1)y + (g - 1)b + (\gamma_G - 1)(g - 1)c$. Since $G_K = I_K H_K$, we can write $g = ih$ and $(g - 1)b = (i - 1)b$. Using $G_K = I_K H_K$ once again, we see that $I_K \rightarrow G_K/H_K$ is surjective, so that we can identify $\gamma_I$ and $\gamma_G$. The resulting cocycle is then cohomologous to $\delta(x)$. This proves that $\delta$ is well-defined.
We now prove that $\delta$ is injective. If $\delta(x) = 0$, then using Lemma A.4 there exists $z \in D(V|I_K)^{\psi=1}$ such that $\delta(x)(\gamma g)$ is the image of $(g - 1)(z)$ in $D(V|I_K)^{\psi=1}/(\gamma I - 1)$. This implies that $(g - 1)(y - z) \in (\gamma I - 1)D(V|I_K)^{\psi=1}$. Applying $\psi - 1$ gives $(g - 1)x = 0$ so that $\tilde{x} \in D(V)^{G_K} \subset V^{I_K} = 0$. The map $\delta$ is therefore injective.

**Lemma A.6.** — If $V$ is semistable and the weights of $V$ are all $\geq 2$, then $\exp_V : D_{dR}(V|I_K) \to H^1(I_K, V)$ is an isomorphism.

Proof. — Apply Thm. 6.8 of [Ber02] to $V|I_K$.

**Proof of Theorem A.1.** — We can replace $K$ by $K_n$ for $n \gg 0$ and use the fact that if $H^2(G_{K_n}, V) = 0$, then $H^2(G_K, V) = 0$ since the restriction map is injective. In particular, we can assume that $H_K I_K = G_K$ and that $\Gamma_K$ is isomorphic to $\Z_p$. By item (2) of Lemma A.2, we have $H^2(G_K, V) = D(V)/(\psi - 1, \gamma - 1)$, and so by Proposition A.5 above, it is enough to prove that

$$H^1(G_K/I_K, (D(V|I_K)/(\gamma I - 1))^{\psi=1}) = 0.$$  

Lemma A.4 tells us that $(D(V|I_K)/(\gamma I - 1))^{\psi=1} = D(V|I_K)^{\psi=1}/(\gamma I - 1)$. Since $D(V|I_K)/(\psi - 1) = 0$ by Lemma A.3, item (1) of Lemma A.2 tells us that $D(V|I_K)^{\psi=1}/(\gamma - 1) = H^1(I_K, V)$.

The map $\exp_V : D_{dR}(V|I_K) \to H^1(I_K, V)$ is an isomorphism by Lemma A.6, and this isomorphism commutes with the action of $G_K$ since it is a natural map. We therefore have $H^1(I_K, V) = \tilde{K}^{ur} \otimes_{K} D_{dR}(V)$ as $G_K$-modules. It remains to observe that the cocycle $\delta(x) \in Z^1(G_K/I_K, \tilde{K}^{ur} \otimes_{K} D_{dR}(V))$ is continuous and that $H^1(G_K/I_K, \tilde{K}^{ur}) = 0$ by taking a lattice, reducing modulo a uniformizer of $K$, and applying Hilbert 90.

**References**


LAURENT BERGER, UMPA – ENS de Lyon, UMR 5669 du CNRS, IUF

E-mail : laurent.berger@ens-lyon.fr  •  URL : perso.ens-lyon.fr/laurent.berger/