THE PERFECTOID COMMUTANT OF LUBIN-TATE POWER SERIES

by

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Abstract. — Let LT be a Lubin-Tate formal group attached to a finite extension of $\mathbf{Q}_p$. By a theorem of Lubin-Sarkis, an invertible characteristic p power series that commutes with the elements of Aut(LT) is itself in Aut(LT). We extend this result to perfectoid power series, by lifting such a power series to characteristic zero and using the theory of locally analytic vectors in certain rings of p-adic periods. This allows us to recover the field of norms of the Lubin-Tate extension from its completed perfection.

Introduction

Let F be a finite extension of $\mathbf{Q}_p$, with ring of integers $\mathcal{O}_F$ and residue field k. Let $q = \mathrm{Card}(k)$ and let π be a uniformizer of $\mathcal{O}_F$. Let LT be the Lubin-Tate formal $\mathcal{O}_F$-module attached to π. Let $F_\infty = F(\mathrm{LT}[\pi^\infty])$ denote the extension of F generated by the torsion points of LT, and let $\Gamma_F = \mathrm{Gal}(F_\infty/F)$. The Lubin-Tate character χ_π gives rise to an isomorphism $\chi_\pi : \Gamma_F \to \mathcal{O}_F^\times$.

The field of norms ([**Win83**]) $\mathbf{E}_F$ of the extension F_∞/F is a local field of characteristic p, endowed with an action of Γ_F, that can be explicitly described as follows. We choose a coordinate T on LT, so that for each $a \in \mathcal{O}_F$ we get a power series $[a](T) \in \mathcal{O}_F[\![T]\!]$. We then have $\mathbf{E}_F = k(\!(Y)\!)$, on which Γ_F acts via the formula $\gamma(f(Y)) = f([\chi_\pi(\gamma)](Y))$. In p-adic Hodge theory, we consider the field $\widetilde{\mathbf{E}}_F$, which is the Y-adic completion of the maximal purely inseparable extension $\cup_{n \geqslant 0} \mathbf{E}_F^{q^{-n}}$ of $\mathbf{E}_F$ inside an algebraic closure. The action of Γ_F extends to the field $\widetilde{\mathbf{E}}_F$. If $f \in \widetilde{\mathbf{E}}_F$ and $\gamma \in \Gamma_F$, we still have $\gamma(f(Y)) = f([\chi_\pi(\gamma)](Y))$. The question that motivated this paper is the following.

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Question. — *Can we recover* $\mathbf{E}_F$ *from the data of the valued field* $\widetilde{\mathbf{E}}_F$ *endowed with the action of* Γ_F*?*

If $a \in \mathcal{O}_F^\times$, then $u(Y) = [a](Y)$ is an element of $\mathbf{E}_F$ of valuation 1 that satisfies the functional equation $u \circ [g](Y) = [g] \circ u(Y)$ for all $g \in \mathcal{O}_F^\times$. Conversely, we prove the following theorem, which answers the question, as it allows us to find a uniformizer of $\mathbf{E}_F$ from the data of the valued field $\widetilde{\mathbf{E}}_F$ endowed with the action of Γ_F.

Theorem A. — *If* $u \in \widetilde{\mathbf{E}}_F$ *is such that* $\mathrm{val}_Y(u) = 1$ *and* $u \circ [g] = [g] \circ u$ *for all* $g \in \mathcal{O}_F^\times$, *then there exists* $a \in \mathcal{O}_F^\times$ *such that* $u(Y) = [a](Y)$.

In particular, $\mathbf{E}_F = k(\!(u)\!)$ for any u as in theorem A. The main difficulty in the proof of theorem A is to prove that if u is as in the statement of theorem A, then there exists $n \geqslant 0$ such that $u \in \mathbf{E}_F^{q^{-n}}$. If $F = \mathbf{Q}_p$ and $\pi = p$, namely in the cyclotomic situation, this follows from the main result of [**BR22**]. However, a crucial ingredient in that paper does not generalize to $F \neq \mathbf{Q}_p$. In order to go beyond the cyclotomic case, we instead use a result of Colmez ([**Col02**]) to lift u to an element $\hat{u}$ of a ring $\tilde{\mathbf{A}}_F^+$ (the Witt vectors over the ring of integers of $\widetilde{\mathbf{E}}_F$, as well as a completion of $\cup_{n \geqslant 0} \varphi_q^{-n}(\mathcal{O}_F[\![\widehat{Y}]\!])$, where $\varphi_q(\widehat{Y}) = [\pi](\widehat{Y})$), that will satisfy a similar functional equation. In particular, $\hat{u}$ is a locally analytic element of a suitable ring of p-adic periods. By previous results of the author ([**Ber16**]), $\hat{u}$ belongs to $\varphi_q^{-n}(\mathcal{O}_F[\![\widehat{Y}]\!])$ for a certain n. This allows us to prove that there exists $n \geqslant 0$ such that $u \in \mathbf{E}_F^{q^{-n}}$. By replacing u with u^{p^k} for a well chosen k, we are led to the study of elements of $Y \cdot k[\![Y]\!]$ under composition. We prove that u is invertible for composition, and to conclude we use a theorem of Lubin-Sarkis ([**LS07**]) saying that if an invertible series commutes with a nontorsion element of Aut(LT), then that series is itself in Aut(LT). We finish this paper with an explanation of why the "Tate traces" on $\widetilde{\mathbf{E}}_F$ used in [**BR22**] don't exist if $F \neq \mathbf{Q}_p$.

1. Locally analytic vectors

We use the notation that was introduced in the introduction. In order to apply lemma 9.3 of [**Col02**], we assume that the coordinate T on LT is chosen such that $[\pi](T)$ is a monic polynomial of degree q (for example, we could ask that $[\pi](T) = T^q + \pi T$).

Let $F_0 = \mathbf{Q}_p^{\mathrm{unr}} \cap F$. Let $\widetilde{\mathbf{E}}_F^+$ denote the ring of integers of $\widetilde{\mathbf{E}}_F$ and let $\tilde{\mathbf{A}}_F^+ = \mathcal{O}_F \otimes_{\mathcal{O}_{F_0}} W(\widetilde{\mathbf{E}}_F^+)$ be the $\mathcal{O}_F$-Witt vectors over $\widetilde{\mathbf{E}}_F^+$.

Proposition 1.1. — *If* $u \in \widetilde{\mathbf{E}}_F^+$ *is such that* $\gamma(u) = [\chi_\pi(\gamma)](u)$ *for all* $\gamma \in \Gamma_F$, *then* u *has a lift* $\hat{u} \in \tilde{\mathbf{A}}_F^+$ *such that* $\gamma(\hat{u}) = [\chi_\pi(\gamma)] \circ \hat{u}$ *for all* $\gamma \in \Gamma_F$.

Proof. — By lemma 9.3 of [**Col02**], there is a unique lift $\hat{u} \in \tilde{\mathbf{A}}_F^+$ of u such that $\varphi_q(\hat{u}) = [\pi](\hat{u})$ (in ibid., this element is denoted by $\{u\}$). If $\gamma \in \Gamma_F$, then both $\gamma(\hat{u})$ and $[\chi_\pi(\gamma)](\hat{u})$ are lifts of u that are compatible with Frobenius as above. By unicity, they are equal. □

Let $\log_{\mathrm{LT}}(T)$ and $\exp_{\mathrm{LT}}(T)$ be the logarithm and exponential series for LT. Write $\exp_{\mathrm{LT}}(T) = \sum_{n \geqslant 1} e_n T^n$ and $\exp_{\mathrm{LT}}(T)^j = \sum_{n \geqslant j} e_{j,n} T^n$ for $j \geqslant 1$.

Lemma 1.2. — *We have* $\mathrm{val}_\pi(e_{j,n}) \geqslant -n/(q-1)$ *for all* $j, n \geqslant 1$.

Proof. — Fix $\varpi \in \overline{\mathbf{Q}}_p$ such that $\mathrm{val}_\pi(\varpi) = 1/(q-1)$ and let $K = F(\varpi)$. Recall that $\log_{\mathrm{LT}}(T) = \lim_{n \to +\infty} [\pi^n](T)/\pi^n$. If $z \in \mathbf{C}_p$ and $\mathrm{val}_\pi(z) \geqslant 1/(q-1)$, then $\mathrm{val}_\pi([\pi](z)) \geqslant \mathrm{val}_\pi(z) + 1$. This implies that $1/\varpi \cdot \log_{\mathrm{LT}}(\varpi T) \in T + T^2 \mathcal{O}_K[\![T]\!]$. Its composition inverse $1/\varpi \cdot \exp_{\mathrm{LT}}(\varpi T)$ therefore also belongs to $T + T^2 \mathcal{O}_K[\![T]\!]$. This implies the claim for $j = 1$. The claim for $j \geqslant 1$ follows easily. □

We use a number of rings of p-adic periods in the Lubin-Tate setting, whose construction and properties were recalled in §3 of [**Ber16**]. Proposition 1.1 gives us an element $\hat{Y} \in \tilde{\mathbf{A}}_F^+$ (denoted by u in ibid.). Let $\widetilde{\mathbf{B}}_F^+ = \tilde{\mathbf{A}}_F^+[1/\pi]$. Given an interval $I = [r; s] \subset [0; +\infty[$, a valuation $V(\cdot, I)$ on $\widetilde{\mathbf{B}}_F^+[1/\hat{Y}]$ is constructed in ibid., as well as various completions of that ring. We use $\widetilde{\mathbf{B}}_F^I$, the completion of $\widetilde{\mathbf{B}}_F^+[1/\hat{Y}]$ for $V(\cdot, I)$ and $\widetilde{\mathbf{B}}_{\mathrm{rig},F}^{\dagger,r} = \varprojlim_{s \geqslant r} \widetilde{\mathbf{B}}_F^{[r;s]}$. Inside $\widetilde{\mathbf{B}}_{\mathrm{rig},F}^{\dagger,r}$, there is the ring $\mathbf{B}_{\mathrm{rig},F}^{\dagger,r}$ of power series $f(\hat{Y})$ with coefficients in F, where $f(T)$ converges on a certain annulus depending on r.

Lemma 1.3. — *If* $s \geqslant 0$*, then* $\mathbf{B}_{\mathrm{rig},F}^{\dagger,s} \cap \tilde{\mathbf{A}}_F^+ = \mathbf{A}_F^+$.

Proof. — Take $f(\hat{Y}) \in \mathbf{B}_{\mathrm{rig},F}^{\dagger,s}$, $t \geqslant s$ and let $I = [s; t]$. We have $V(f, I) \geqslant 0$, so that f is bounded by 1 on the corresponding annulus. This is true for all t, so that $f \in \mathbf{B}_F^{\dagger,s}$. We now have $f \in \mathbf{B}_F^{\dagger,s} \cap \tilde{\mathbf{A}}_F^+ = \mathbf{A}_F^+$. □

Let W be a Banach space with a continuous action of Γ_F. The notion of locally analytic vector was introduced in [**ST03**]. Recall (see for instance §2 of [**Ber16**]; the definition given there is easily seen to be equivalent to the following one) that an element $w \in W$ is locally F-analytic if there exists a sequence $\{w_k\}_{k \geqslant 0}$ of W such that $w_k \to 0$, and an integer $n \geqslant 1$ such that for all $\gamma \in \Gamma_F$ such that $\chi_\pi(\gamma) = 1 + p^n c(\gamma)$ with $c(\gamma) \in \mathcal{O}_F$, we have $\gamma(w) = \sum_{k \geqslant 0} c(\gamma)^k w_k$. If $W = \varprojlim_i W_i$ is a Fréchet representation of Γ_F, we say that $w \in W$ is pro-F-analytic if its image in W_i is locally F-analytic for all i.

Proposition 1.4. — *If* $r \geqslant 0$ *and* $x \in \tilde{\mathbf{A}}_F^+$ *is such that* $\mathrm{val}_Y(\overline{x}) > 0$ *and* $\gamma(x) = [\chi_\pi(\gamma)](x)$ *for all* $\gamma \in \Gamma_F$*, then* x *is a pro-*F*-analytic element of* $\widetilde{\mathbf{B}}_{\mathrm{rig},F}^{\dagger,r}$.

Proof. — We prove that for all $s \geqslant r$, x is a locally F-analytic vector of $\widetilde{\mathbf{B}}_F^{[r;s]}$. The proposition then follows, since $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},F} = \varprojlim_{s \geqslant r} \widetilde{\mathbf{B}}_F^{[r;s]}$ as Fréchet spaces.

Let $S(X,Y) = \sum_{i,j} s_{i,j} X^i Y^j \in \mathcal{O}_F[\![X,Y]\!]$ be the power series that gives the addition in LT. We have $\log_{\mathrm{LT}}(x) \in \widetilde{\mathbf{B}}_F^{[r;s]}$. Take $n \geqslant 1$ such that $V(p^{n-1}\log_{\mathrm{LT}}(x),[r;s]) > 0$. We have $[a](T) = \exp_{\mathrm{LT}}(a \log_{\mathrm{LT}}(T))$, so that $[1+p^n c](T) = S(T, \exp_{\mathrm{LT}}(p^n c \log_{\mathrm{LT}}(T)))$. If $\chi_\pi(\gamma) = 1 + p^n c(\gamma)$, then

$$\begin{aligned}\gamma(x) &= \sum_{k \geqslant 0} c(\gamma)^k \sum_{j \leqslant k} p^{nk} e_{j,k} \log_{\mathrm{LT}}(x)^k \sum_{i \geqslant 0} s_{i,j} x^i \\ &= \sum_{k \geqslant 0} c(\gamma)^k \sum_{j \leqslant k} p^k e_{j,k} \cdot (p^{n-1}\log_{\mathrm{LT}}(x))^k \cdot \sum_{i \geqslant 0} s_{i,j} x^i.\end{aligned}$$

We have $p^k e_{j,k} \in \mathcal{O}_F$ by lemma 1.2, $V(p^{n-1}\log_{\mathrm{LT}}(x),[r;s]) > 0$ by hypothesis, $s_{i,j} \in \mathcal{O}_F$ and $V(x,[r;s]) > 0$. This implies the claim. □

Proposition 1.5. — *If $r > 0$ and $x \in \tilde{\mathbf{A}}_F^+$ is a pro-F-analytic element of $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},F}$, then there exists $n \geqslant 0$ such that $x \in \varphi_q^{-n}(\mathbf{A}_F^+)$.*

Proof. — By item (3) of theorem 4.4 of [**Ber16**] (applied with $K = F$), there exists $n \geqslant 0$ and $s > 0$ such that $x \in \varphi_q^{-n}(\mathbf{B}^{\dagger,s}_{\mathrm{rig},F})$. The proposition now follows from lemma 1.3 applied to $\varphi_q^n(x)$. □

2. Composition of power series

Recall that a power series $f(Y) \in k[\![Y]\!]$ is separable if $f'(Y) \neq 0$. If $f(Y) \in Y \cdot k[\![Y]\!]$, we say that f is invertible if $f'(0) \in k^\times$, which is equivalent to f being invertible for composition (denoted by $\circ$). We say that $w(Y) \in Y \cdot k[\![Y]\!]$ is nontorsion if $w^{\circ n}(Y) \neq Y$ for all $n \geqslant 1$. If $w(Y) = \sum_{i \geqslant 0} w_i Y^i \in k[\![Y]\!]$ and $m \in \mathbf{Z}$, let $w^{(m)}(Y) = \sum_{i \geqslant 0} w_i^{p^m} Y^i$. Note that $(w \circ v)^{(m)} = w^{(m)} \circ v^{(m)}$.

Proposition 2.1. — *Let $w(Y) \in Y + Y^2 \cdot k[\![Y]\!]$ be an invertible nontorsion series, and let $f(Y) \in Y \cdot k[\![Y]\!]$ be a separable power series. If $w^{(m)} \circ f = f \circ w$, then f is invertible.*

Proof. — This is a slight generalization of lemma 6.2 of [**Lub94**]. Write

$$\begin{aligned} f(Y) &= f_n Y^n + \mathrm{O}(Y^{n+1}) \\ f'(Y) &= g_k Y^k + \mathrm{O}(Y^{k+1}) \\ w(Y) &= Y + w_r Y^r + \mathrm{O}(Y^{r+1}),\end{aligned}$$

with $f_n, g_k, w_r \neq 0$. Since w is nontorsion, we can replace w by $w^{\circ p^\ell}$ for $\ell \gg 0$ and assume that $r \geqslant k+1$. We have

$$\begin{aligned} w^{(m)} \circ f &= f(Y) + w_r^{(m)} f(Y)^r + \mathrm{O}(Y^{n(r+1)}) \\ &= f(Y) + w_r^{(m)} f_n^r Y^{nr} + \mathrm{O}(Y^{nr+1}). \end{aligned}$$

If $k=0$, then $n=1$ and we are done, so assume that $k \geqslant 1$. We have

$$\begin{aligned} f \circ w &= f(Y + w_r Y^r + \mathrm{O}(Y^{r+1})) \\ &= f(Y) + w_r Y^r f'(Y) + \mathrm{O}(Y^{2r}) \\ &= f(Y) + w_r g_k Y^{r+k} + \mathrm{O}(Y^{r+k+1}). \end{aligned}$$

This implies that $nr = r+k$, hence $(n-1)r = k$, which is impossible if $r > k$ unless $n=1$. Hence $n=1$ and f is invertible. $\square$

We now prove theorem A. Take $u \in \widetilde{\mathbf{E}}_F$ such that $\mathrm{val}_Y(u) = 1$ and $u \circ [g] = [g] \circ u$ for all $g \in \mathcal{O}_F^\times$. By proposition 1.1, u has a lift $\hat{u} \in \tilde{\mathbf{A}}_F^+$ such that $\gamma(\hat{u}) = [\chi_\pi(\gamma)] \circ \hat{u}$ for all $\gamma \in \Gamma_F$. By proposition 1.4, $\hat{u}$ is a pro-F-analytic element of $\widetilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},F}$. By proposition 1.5, there exists $n \geqslant 0$ such that $\hat{u} \in \varphi_q^{-n}(\mathbf{A}_F^+)$. This implies that $u \in \varphi_q^{-n}(\mathbf{E}_F^+)$. Hence there is an $m \in \mathbf{Z}$ such that $f(Y) = u(Y)^{p^m}$ belongs to $Y \cdot k[\![Y]\!]$ and is separable. Note that $\mathrm{val}_Y(f) = p^m$. Take $g \in 1 + \pi\mathcal{O}_F$ such that g is nontorsion, and let $w(Y) = [g](Y)$ so that $u \circ w = w \circ u$. We have $f \circ w = w^{(m)} \circ f$. By proposition 2.1, f is invertible. This implies that $\mathrm{val}_Y(f) = 1$, so that $m = 0$ and u itself is invertible. Since $u \circ [g] = [g] \circ u$ for all $g \in \mathcal{O}_F^\times$, theorem 6 of [**LS07**] implies that $u \in \mathrm{Aut}(\mathrm{LT})$. Hence there exists $a \in \mathcal{O}_F^\times$ such that $u(Y) = [a](Y)$.

3. Tate traces in the Lubin-Tate setting

If $F = \mathbf{Q}_p$ and $\pi = p$ (namely in the cyclotomic situation) the fact that, in the proof of theorem A, there exists $n \geqslant 0$ such that $u \in \varphi_q^{-n}(\mathbf{E}_F^+)$ follows from the main result of [**BR22**]. We now explain why the methods of ibid don't extend to the Lubin-Tate case. More precisely, we prove that there is no Γ_F-equivariant k-linear projector $\widetilde{\mathbf{E}}_F \to \mathbf{E}_F$ if $F \neq \mathbf{Q}_p$. Choose a coordinate T on LT such that $\log_{\mathrm{LT}}(T) = \sum_{n \geqslant 0} T^{q^n}/\pi^n$, so that $\log'_{\mathrm{LT}}(T) \equiv 1 \bmod \pi$. Let $\partial = 1/\log'_{\mathrm{LT}}(T) \cdot d/dT$ be the invariant derivative on LT.

Lemma 3.1. — *We have $d\gamma(Y)/dY \equiv \chi_\pi(\gamma)$ in $\mathbf{E}_F$ for all $\gamma \in \Gamma_F$.*

Proof. — Since $\log'_{\mathrm{LT}} \equiv 1 \bmod \pi$, we have $\partial = d/dY$ in $\mathbf{E}_F$. Applying $\partial \circ \gamma = \chi_\pi(\gamma)\gamma \circ \partial$ to Y, we get the claim. $\square$

Lemma 3.2. — *If $\gamma \in \Gamma_F$ is nontorsion, then $\mathbf{E}_F^{\gamma=1} = k$.*

Proposition 3.3. — *If $F \neq \mathbf{Q}_p$, there is no Γ_F-equivariant map $R : \mathbf{E}_F \to \mathbf{E}_F$ such that $R(\varphi_q(f)) = f$ for all $f \in \mathbf{E}_F$.*

Proof. — Suppose that such a map exists, and take $\gamma \in \Gamma_F$ nontorsion and such that $\chi_\pi(\gamma) \equiv 1 \bmod \pi$. We first show that if $f \in \mathbf{E}_F$ is such that $(1-\gamma)f \in \varphi_q(\mathbf{E}_F)$, then $f \in \varphi_q(\mathbf{E}_F)$. Write $f = f_0 + \varphi_q(R(f))$ where $f_0 = f - \varphi_q(R(f))$, so that $R(f_0) = 0$ and $(1-\gamma)f_0 = \varphi_q(g) \in \varphi_q(\mathbf{E}_F)$. Applying R, we get $0 = (1-\gamma)R(f_0) = g$. Hence $g = 0$ so that $(1-\gamma)f_0 = 0$. Since $\mathbf{E}_F^{\gamma=1} = k$ by lemma 3.2, this implies $f_0 \in k$, so that $f \in \varphi_q(\mathbf{E}_F)$.

However, lemma 3.1 and the fact that $\chi_\pi(\gamma) \equiv 1 \bmod \pi$ imply that $\gamma(Y) = Y + f_\gamma(Y^p)$ for some $f_\gamma \in \mathbf{E}_F$, so that $\gamma(Y^{q/p}) = Y^{q/p} + \varphi_q(g_\gamma)$. Hence $(1-\gamma)(Y^{q/p}) \in \varphi_q(\mathbf{E}_F)$ even though $Y^{q/p}$ does not belong to $\varphi_q(\mathbf{E}_F)$. Therefore, no such map R can exist. □

Corollary 3.4. — *If $F \neq \mathbf{Q}_p$, there is no Γ_F-equivariant k-linear projector $\varphi_q^{-1}(\mathbf{E}_F) \to \mathbf{E}_F$. A fortiori, there is no Γ_F-equivariant k-linear projector $\widetilde{\mathbf{E}}_F \to \mathbf{E}_F$.*

Proof. — Given such a projector T, we could define R as in prop 3.3 by $R = T \circ \varphi_q^{-1}$. □

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