# ON THE REDUCTION MODULO $p$ OF SOME CRYSTALLINE REPRESENTATIONS

*by*

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## Introduction

The goal of this talk was to recall part of Breuil's "mod $p$" and "continuous" $p$-adic Langlands correspondences (in the supersingular case), and to explain the proof of the conjecture relating the two. Note : most relevant articles are in the bibliographies of the three articles given as references, the first two of which contain the proofs of the two main results discussed in the lecture.

## 1. Objects in characteristic $p$

On the $\mathrm{GL}_2(\mathbf{Q}_p)$-side, the objects which we're interested in are the smooth irreducible $\overline{\mathbf{F}}_p$-representations of $\mathrm{GL}_2(\mathbf{Q}_p)$ having a central character. Choose $r \in \{0, \cdots, p-1\}$ and let $\chi$ be a smooth character of $\mathbf{Q}_p^\times$. Define

$$\pi(r,\chi) := \left[\left(\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^r \overline{\mathbf{F}}_p^2\right)/T\right] \otimes (\chi \circ \det),$$

where $T$ is a Hecke operator defined by Barthel and Livné. The representations $\pi(r,\chi)$ are called supersingular, and any smooth irreducible $\overline{\mathbf{F}}_p$-representation of $\mathrm{GL}_2(\mathbf{Q}_p)$ having a central character is either one-dimensional, a special series, a principal series or supersingular. Furthermore, the intertwinings between the $\pi(r,\chi)$'s are given by

$$\pi(r,\chi) = \pi(p-1-r, \chi\omega^r) = \pi(p-1-r, \chi\omega^r\mu_{-1}) = \pi(r, \chi\mu_{-1}),$$

where $\omega$ is the mod $p$ cyclotomic character and $\mu_\lambda$ is the unramified character of $\mathbf{Q}_p^\times$ sending $p$ to $\lambda$.

On the Galois side, it is easy to classify the irreducible 2-dimensional $\overline{\mathbf{F}}_p$-representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Let $\omega_2$ be Serre's fundamental character of level 2, and for $s \in \{1, \cdots, p\}$ let $\mathrm{ind}(\omega_2^s)$ denote the unique irreducible 2-dimensional $\overline{\mathbf{F}}_p$-representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ whose determinant is $\omega^s$ and whose restriction to inertia is $\omega_2^s \oplus \omega_2^{ps}$. If $r \in \{0, \cdots, p-1\}$

and if $\chi$ is a continuous character of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, let $\rho(r,\chi) := \mathrm{ind}(\omega_2^{r+1}) \otimes \chi$. Any irreducible 2-dimensional $\overline{\mathbf{F}}_p$-representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ is isomorphic to some such $\rho(r,\chi)$ and the intertwinings between the $\rho(r,\chi)$'s are given by

$$\rho(r,\chi) = \rho(p-1-r, \chi\omega^r) = \rho(p-1-r, \chi\omega^r\mu_{-1}) = \rho(r, \chi\mu_{-1}).$$

It is therefore natural to define a correspondence : $\pi(r,\chi) \leftrightarrow \rho(r,\chi)$. Breuil has also defined a correspondence between special & principal series and reducible representations but this was not covered in the talk due to lack of time.

## 2. Objects in characteristic 0

Let $E$ be a finite extension of $\mathbf{Q}_p$, choose $k \geq 2$ and $a_p \in E$ such that $\mathrm{val}(a_p) > 0$.

We start by defining the objects on the Galois side, they are 2-dimensional crystalline representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Let $D_{k,a_p}$ be the filtered $\varphi$-module $D_{k,a_p} := Ee_1 \oplus Ee_2$ where

$$\begin{cases} \varphi(e_1) = p^{k-1}e_2 \\ \varphi(e_2) = -e_1 + a_p e_2 \end{cases} \quad \text{and} \quad \mathrm{Fil}^i D_{k,a_p} = \begin{cases} D_{k,a_p} & \text{if } i \leq 0, \\ E \cdot e_1 & \text{if } 1 \leq i \leq k-1, \\ 0 & \text{if } i \geq k. \end{cases}$$

We define $V_{k,a_p}$ as the crystalline representation such that $\mathrm{D}_{\mathrm{cris}}(V_{k,a_p}^*) = D_{k,a_p}$ (note that then $V_{k,a_p}^* = V_{k,a_p}(1-k)$) so that $V_{k,a_p}$ is a crystalline representation whose Hodge-Tate weights are 0 and $k-1$. Choose a $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-stable lattice $T$ of $V_{k,a_p}$ and set $\overline{V}_{k,a_p} := (\overline{\mathbf{F}}_p \otimes_{O_E} T)^{\mathrm{ss}}$. This representation does not depend on the choice of $T$ by Brauer-Nesbitt.

We now move to the $\mathrm{GL}_2(\mathbf{Q}_p)$-side, and we define the following locally algebraic representation :

$$\Pi_{k,a_p} := \left(\mathrm{ind}_{\mathrm{GL}_2(\mathbf{Z}_p)\mathbf{Q}_p^\times}^{\mathrm{GL}_2(\mathbf{Q}_p)} \mathrm{Sym}^{k-2}\overline{\mathbf{Q}}_p^2\right) / (T - a_p).$$

Breuil has studied this representation and conjectured the existence of a $\overline{\mathbf{Z}}_p$-lattice. This is now a theorem (see [**1**] – and note that for simplicity, we assumed that $\varphi$ is semi-simple on $D_{k,a_p}$) and we then set $\overline{\Pi}_{k,a_p} := [\overline{\mathbf{F}}_p \otimes_{\overline{\mathbf{Z}}_p} (\text{this lattice})]^{\mathrm{ss}}$. This does not depend on the choice of lattice by Brauer-Nesbitt. Breuil's conjecture regarding the compatibility of the correspondences $\Pi_{k,a_p} \leftrightarrow V_{k,a_p}$ and $\pi(r,\chi) \leftrightarrow \rho(r,\chi)$ is then the following :

**Conjecture** — We have $\overline{\Pi}_{k,a_p} = \pi(r,\chi)$ if and only if $\overline{V}_{k,a_p} = \rho(r,\chi)$.

Note that there is also a conjecture concerning the "reducible" cases. The above conjecture is now a theorem (see [**2**]); it was first proved for $k \leq p$ by Breuil, who also computed $\overline{\Pi}_{k,a_p}$ for $k \leq 2p+2$.

## 3. A new model for $\Pi_{k,a_p}$

We start by explaining the proof of the existence of a lattice in $\Pi_{k,a_p}$. The idea of the proof (based on a similar construction carried out by Colmez in the semi-stable case) is to construct a Banach representation $\Pi(V_{k,a_p})$ of $\mathrm{GL}_2(\mathbf{Q}_p)$ with a map $\Pi_{k,a_p} \to \Pi(V_{k,a_p})$. One then uses the theory of $(\varphi,\Gamma)$-modules to prove that $\Pi(V_{k,a_p}) \neq 0$ and is irreducible (and admissible) which shows the existence of a lattice of $\Pi_{k,a_p}$ (the map $\Pi_{k,a_p} \to \Pi(V_{k,a_p})$ being injective with dense image).

Let $\mathcal{O} := \{\sum_{i\in\mathbf{Z}} a_i X^i$, where $a_i \in O_E$ and $a_i \to 0$ as $i \to -\infty\}$. We endow this ring with a frobenius $\varphi$ such that $\varphi(X) = (1+X)^p - 1$ and an action of $\Gamma := \mathbf{Z}_p^\times$ such that $[a]\cdot X = (1+X)^a - 1$. A $(\varphi,\Gamma)$-module is an $\mathcal{O}$-module $\mathbf{D}$ of finite type, endowed with a frobenius $\varphi$ and an action of $\Gamma$ which are continuous, semi-linear, commute with each other, and such that $\varphi^*(\mathbf{D}) = \mathbf{D}$. Recall that Fontaine constructed an equivalence of categories between the category of $O_E$-representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and the category of $(\varphi,\Gamma)$-modules. If $V = E \otimes_{O_E} T$ is an $E$-linear representation, we'll call $\mathbf{D}(V) := E \otimes_{O_E} \mathbf{D}(T)$ the associated $(\varphi,\Gamma)$-module.

One can show that if $\mathbf{D}$ is a $(\varphi,\Gamma)$-module and $y \in \mathbf{D}$, then there exist uniquely determined elements $y_0, \cdots, y_{p-1}$ such that $y = \sum_{i=0}^{p-1}(1+X)^i\varphi(y_i)$ and we define $\psi(y) := y_0$, which makes $\psi$ into a $\Gamma$-equivariant left inverse of $\varphi$. Let $V_{k,a_p}$ be the representation defined in the previous section and let $(\varprojlim_\psi \mathbf{D}(V_{k,a_p}))^b$ denote the set of sequences $(v_0, v_1, \cdots)$ such that $\psi(v_i) = v_{i-1}$ and such that the sequence $\{v_0, v_1, \cdots\}$ is bounded for the "weak topology" (which one can think of as the $(p,X)$-adic topology). We make $(\varprojlim_\psi \mathbf{D}(V_{k,a_p}))^b$ into a representation of the Borel subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$ in the following way; if $v \in (\varprojlim_\psi \mathbf{D}(V_{k,a_p}))^b$, then :

$$\begin{aligned}
\left(\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\cdot v\right)_n &= x_0^{k-2} v_n \text{ where } x = p^{\mathrm{val}(x)} x_0; \\
\left(\begin{pmatrix} 1 & 0 \\ 0 & p^j \end{pmatrix}\cdot v\right)_n &= v_{n-j} = \psi^j(v_n); \\
\left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\cdot v\right)_n &= [a](v_n); \\
\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\cdot v\right)_n &= \psi^m((1+X)^{p^{n+m}z} v_{n+m}),\ n+m \geq -\mathrm{val}(z).
\end{aligned}$$

The Banach space $\Pi(V_{k,a_p})$ alluded to above can be shown to be isomorphic to another Banach space of "class $\mathcal{C}^r$ functions" (for an $r$ depending on $k$ and $a_p$) on $\mathbf{P}^1$ satisfying a number of conditions, together with an action of $\mathrm{GL}_2(\mathbf{Q}_p)$. Using ideas of Colmez, we can interpret elements of $(\varprojlim_\psi \mathbf{D}(V_{k,a_p}))^b$ as distributions on $\mathbf{P}^1$, i.e. elements of the dual of this new Banach space and thus we have $\Pi(V_{k,a_p})^* \simeq (\varprojlim_\psi \mathbf{D}(V_{k,a_p}))^b$ as representations of the Borel subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$. A classical result allows one to interpret $\Pi_{k,a_p}$ as a space of locally polynomial functions on $\mathbf{P}^1$ satisfying similar conditions as those used to define $\Pi(V_{k,a_p})$ and this explains why $\Pi_{k,a_p}$ injects into $\Pi(V_{k,a_p})$ (the image is dense because $\Pi(V_{k,a_p})$ is irreducible).

## 4. Proof of the conjecture

Using the isomorphism $\Pi(V_{k,a_p})^* \simeq (\varprojlim_\psi \mathbf{D}(V_{k,a_p}))^b$, one can show (see [**3**]) that $\overline{\Pi}^*_{k,a_p} \simeq (\varprojlim_\psi \mathbf{D}(\overline{V}_{k,a_p}))^b$ and this already proves one implication in the conjecture. Now suppose that $\overline{\Pi}_{k,a_p} = \pi(r,\chi)$ for some $r,\chi$. By examination, we see that there exists $k' \in \{2,\cdots,p\}$ such that $\pi(r,\chi) = \overline{\Pi}_{k',0}$ and the conjecture in this case results from computations of Breuil. In particular, $\overline{V}_{k',0} \simeq \rho(r,\chi)$ on the one hand and $\overline{\Pi}^*_{k',0} \simeq (\varprojlim_\psi \mathbf{D}(\overline{V}_{k',0}))^b$ on the other hand so that $(\varprojlim_\psi \mathbf{D}(\overline{V}_{k,a_p}))^b \simeq (\varprojlim_\psi \mathbf{D}(\overline{V}_{k',0}))^b$ and using the theory of $(\varphi,\Gamma)$-modules it is not hard to show that $\overline{V}_{k,a_p} = \overline{V}_{k',0}$ so that $\overline{V}_{k,a_p} = \rho(r,\chi)$. This proves the other implication in the conjecture.

## References


[1] L. Berger, C. Breuil – *Représentations cristallines irréductibles de* $\mathrm{GL}_2(\mathbf{Q}_p)$, preprint.

[2] L. Berger, C. Breuil – *Sur la réduction des représentations cristallines de dimension* 2 *en poids moyens*, preprint.

[3] P. Colmez – *Une correspondance de Langlands locale p-adique pour les représentations semi-stables de dimension* 2, preprint.


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