B-PAIRS AND ($\varphi, \Gamma$)-MODULES

by

Laurent Berger

The goal of the talk was to present some of the results from my article [1]. Let $K$ be a $p$-adic base field, for example some finite extension of $\mathbb{Q}_p$. One of the aims of $p$-adic Hodge theory is to describe some of the $p$-adic representations of $G_K = \text{Gal}(\overline{K}/K)$, namely those which “come from geometry”, in terms of some more amenable objects. The most satisfying result in this direction is Colmez-Fontaine’s theorem which states that the functor $V \mapsto D_{st}(V)$ gives rise to an equivalence of categories between the category of semistable $p$-adic representations and the category of admissible filtered ($\varphi, N$)-modules.

If $D$ is a filtered ($\varphi, N$)-module coming from the cohomology of a scheme $X$, then the underlying ($\varphi, N$)-module only depends on the special fiber of $X$ (it is its log-crystalline cohomology) and the filtration only depends on the generic fiber of $X$ (it is its de Rham cohomology). If $D_1$ and $D_2$ are two filtered ($\varphi, N$)-modules and $B_e = B_{\text{cris}}^1$ then the ($\varphi, N$)-modules $D_1$ and $D_2$ are isomorphic if and only if $(B_{\text{st}} \otimes_{K_0} D_1)^N = 0, \varphi = 1$ and $(B_{\text{st}} \otimes_{K_0} D_2)^N = 0, \varphi = 1$ are isomorphic as $B_e$-representations of $G_K$. Similarly, the filtered modules $K \otimes_{K_0} D_1$ and $K \otimes_{K_0} D_2$ are isomorphic if and only if $\text{Fil}^0(B_{\text{dR}} \otimes_{K_0} D_1)$ and $\text{Fil}^0(B_{\text{dR}} \otimes_{K_0} D_2)$ are isomorphic as $B_{\text{dR}}^+$-representations of $G_K$.

The main idea of [1] is to separate the phenomena related to the special fiber from those related to the generic fiber by considering not just $p$-adic representations but $B$-pairs $W = (W_e, W_{dR}^+)$ where $W_e$ is a $B_e$-representation of $G_K$ and $W_{dR}^+$ is a $B_{dR}^+$-representation of $G_K$ and $B_{dR} \otimes_{B_e} W_e = B_{dR} \otimes_{B_{dR}^+} W_{dR}^+$. If $V$ is a $p$-adic representation, then one associates to it $W(V) = (B_e \otimes_{\mathbb{Q}_p} V, B_{dR}^+ \otimes_{\mathbb{Q}_p} V)$ and this defines a fully faithful functor from the category of $p$-adic representations to the category of $B$-pairs. One can extend the usual definitions of $p$-adic Hodge theory from $p$-adic representations to all $B$-pairs. For example, we say that a $B$-pair $W$ is semistable if $B_{\text{st}} \otimes_{B_e} W_e$ is trivial and it is easy to see that the functor $D \mapsto W(D)$ which to a filtered ($\varphi, N$)-module $D$ assigns the semistable $B$-pair $W(D) = ((B_{\text{st}} \otimes_{K_0} D)^N, \varphi = 1, \text{Fil}^0(B_{\text{dR}} \otimes_{K_0} D))$ is an equivalence of categories.
One of the main general purpose tools which we have for studying $p$-adic representations is the theory of $(\varphi, \Gamma)$-modules. There is an equivalence of categories between the category of $p$-adic representations and the category of étale $(\varphi, \Gamma)$-modules over the Robba ring. The main result of [1] is that one can associate to every $B$-pair $W$ a $(\varphi, \Gamma)$-module $D(W)$ over the Robba ring and that the resulting functor is then an equivalence of categories.

The article [1] includes some other results which were not discussed in the lecture, among which: a description of isoclinic $(\varphi, \Gamma)$-modules, an answer to a question of Fontaine regarding $B_{\text{cris}}^{\varphi=1}$-representations, and a description of finite height $(\varphi, \Gamma)$-modules.

References