B-PAIRS AND (φ, Γ)-MODULES

by

Laurent Berger

The goal of the talk was to present some of the results from my article [**1**]. Let K be a p-adic base field, for example some finite extension of $\mathbf{Q}_p$. One of the aims of p-adic Hodge theory is to describe some of the p-adic representations of $G_K = \mathrm{Gal}(\overline{K}/K)$, namely those which "come from geometry", in terms of some more amenable objects. The most satisfying result in this direction is Colmez-Fontaine's theorem which states that the functor $V \mapsto \mathrm{D}_{\mathrm{st}}(V)$ gives rise to an equivalence of categories between the category of semistable p-adic representations and the category of admissible filtered (φ, N)-modules.

If D is a filtered (φ, N)-module coming from the cohomology of a scheme X, then the underlying (φ, N)-module only depends on the special fiber of X (it is its log-crystalline cohomology) and the filtration only depends on the generic fiber of X (it is its de Rham cohomology). If D_1 and D_2 are two filtered (φ, N)-modules and $\mathbf{B}_{\mathrm{e}} = \mathbf{B}_{\mathrm{cris}}^{\varphi=1}$ then the (φ, N)-modules D_1 and D_2 are isomorphic if and only if $(\mathbf{B}_{\mathrm{st}} \otimes_{K_0} D_1)^{N=0,\varphi=1}$ and $(\mathbf{B}_{\mathrm{st}} \otimes_{K_0} D_2)^{N=0,\varphi=1}$ are isomorphic as $\mathbf{B}_{\mathrm{e}}$-representations of G_K. Similarly, the filtered modules $K \otimes_{K_0} D_1$ and $K \otimes_{K_0} D_2$ are isomorphic if and only if $\mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_{K_0} D_1)$ and $\mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_{K_0} D_2)$ are isomorphic as $\mathbf{B}_{\mathrm{dR}}^+$-representations of G_K.

The main idea of [**1**] is to separate the phenomena related to the special fiber from those related to the generic fiber by considering not just p-adic representations but *B-pairs* $W = (W_e, W_{dR}^+)$ where W_e is a $\mathbf{B}_{\mathrm{e}}$-representation of G_K and W_{dR}^+ is a $\mathbf{B}_{\mathrm{dR}}^+$-representation of G_K and $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{e}}} W_e = \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{dR}}^+} W_{dR}^+$. If V is a p-adic representation, then one associates to it $W(V) = (\mathbf{B}_{\mathrm{e}} \otimes_{\mathbf{Q}_p} V, \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbf{Q}_p} V)$ and this defines a fully faithful functor from the category of p-adic representations to the category of B-pairs. One can extend the usual definitions of p-adic Hodge theory from p-adic representations to all B-pairs. For example, we say that a B-pair W is semistable if $\mathbf{B}_{\mathrm{st}} \otimes_{\mathbf{B}_{\mathrm{e}}} W_e$ is trivial and it is easy to see that the functor $D \mapsto W(D)$ which to a filtered (φ, N)-module D assigns the semistable B-pair $W(D) = ((\mathbf{B}_{\mathrm{st}} \otimes_{K_0} D)^{N=0,\varphi=1}, \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_{K_0} D))$ is an equivalence of categories.

One of the main general purpose tools which we have for studying p-adic representations is the theory of (φ, Γ)-modules. There is an equivalence of categories between the category of p-adic representations and the category of étale (φ, Γ)-modules over the Robba ring. The main result of [**1**] is that one can associate to every B-pair W a (φ, Γ)-module $D(W)$ over the Robba ring and that the resulting functor is then an equivalence of categories.

The article [**1**] includes some other results which were not discussed in the lecture, among which : a description of isoclinic (φ, Γ)-modules, an answer to a question of Fontaine regarding $\mathbf{B}_{\mathrm{cris}}^{\varphi=1}$-representations, and a description of finite height (φ, Γ)-modules.

References

[1] L. Berger, *B-paires et (φ, Γ)-modules*, Preprint, avril 2007.

Laurent Berger