TRIANGULINE REPRESENTATIONS

by

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Trianguline representations are a special class of p-adic representations. Let K be a finite extension of $\mathbf{Q}_p$ and let $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$. Fontaine has extensively studied p-adic representations (finite dimensional E-linear representations of G_K where E, the field of coefficients, is a finite extension of $\mathbf{Q}_p$). In particular, he has defined the important and useful notions of de Rham, semistable and crystalline representations. Trianguline representations have been defined by Colmez in the course of his work on the p-adic Langlands correspondence of Breuil. His definition is in terms of (φ, Γ)-modules over the Robba ring and we give it here in the case $K = \mathbf{Q}_p$ in order to simplify the notation.

Let $\mathcal{R} = \{f(X) = \sum_{n \in \mathbf{Z}} a_n X^n$ where $a_n \in E$ and there exists $\rho(f)$ such that $f(X)$ converges for $\rho(f) < |X|_p < 1\}$ be the Robba ring. The ring $\mathcal{E}^\dagger$ is the subring of $\mathcal{R}$ consisting of bounded power series and $\mathcal{O}_{\mathcal{E}}^\dagger$ is the set of $f(X) \in \mathcal{R}$ with $|a_n|_p \leq 1$ for all n. All of those rings are endowed with a frobenius φ given by $\varphi(f)(X) = f((1+X)^p - 1)$ and an action of the group $\Gamma \simeq \mathbf{Z}_p^\times$ given by $[a](f)(X) = f((1+X)^a - 1)$ where $[\cdot] : \mathbf{Z}_p^\times \to \Gamma$ denotes the isomorphism between $\mathbf{Z}_p^\times$ and Γ.

A (φ, Γ)-module is a free $\mathcal{R}$-module of finite rank d endowed with a semilinear frobenius φ such that $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathcal{R})$ and with a commuting semilinear continuous action of Γ. We say that such an object is étale if there exists a basis in which $\mathrm{Mat}(\varphi) \in \mathrm{GL}_d(\mathcal{O}_{\mathcal{E}}^\dagger)$.

The main result relating (φ, Γ)-modules and p-adic Galois representations is the following (it combines theorems of Fontaine, Fontaine-Wintenberger, Cherbonnier-Colmez and Kedlaya) : if D is an étale (φ, Γ)-module, and if $\widetilde{\mathcal{R}}$ denotes one of Fontaine's rings, then $V = (\widetilde{\mathcal{R}} \otimes_{\mathcal{R}} \mathrm{D})^{\varphi=1}$ is a p-adic representation and the resulting functor gives rise to an equivalence of categories : $\{$étale (φ, Γ)-modules$\} \to \{p$-adic representations$\}$. In this way, one realizes the category of p-adic representations as a full subcategory of a larger one, the category of all (φ, Γ)-modules over $\mathcal{R}$.

We then say that a (φ, Γ)-module D is triangulable if it is an iterated extension of objects of rank 1, that is if we can write $0 = \mathrm{D}_0 \subset \mathrm{D}_1 \subset \cdots \subset \mathrm{D}_\ell = \mathrm{D}$ where each D_i

is a (φ, Γ)-module and $\mathrm{D}_i/\mathrm{D}_{i-1}$ is of rank 1. If V is a p-adic representation, then we say that it is split-trianguline if the associated (φ, Γ)-module is triangulable, and we say it is trianguline if there exists some finite extension F/E such that $F \otimes_E V$ is split-trianguline.

Examples of trianguline representations include all semi-stable representations and also the representations associated to finite slope overconvergent modular forms (by a theorem of Kisin). In particular, trianguline representations are an important tool in the study of eigencurves and eigenvarieties, as in the work of Bellaïche and Chenevier. They are also used by Colmez (and were defined for that purpose) in his construction of the "unitary principal series of $\mathrm{GL}_2(\mathbf{Q}_p)$" which realizes Breuil's p-adic Langlands correspondence for trianguline representations.

In order to classify trianguline representations, one needs a classification of rank 1 (φ, Γ)-modules as well as the knowledge of the associated Ext^1 groups. If $\delta : \mathbf{Q}_p^\times \to E$ is a continuous character, one defines the (φ, Γ)-module $\mathcal{R}(\delta) = \mathcal{R} \cdot e_\delta$ where $\varphi(e_\delta) = \delta(p)e_\delta$ and $[a](e_\delta) = \delta(a)e_\delta$. It is then a result of Colmez that every (φ, Γ)-module of rank 1 is isomorphic to a $\mathcal{R}(\delta)$ for a well-defined δ. Note that one can define the slope of $\mathcal{R}(\delta)$ to be $\mathrm{val}_p(\delta(p))$ and the weight of $\mathcal{R}(\delta)$ to be $\lim_{a \to 1} \log_p \delta(a)/\log_p(a)$. In addition, although I have not defined (φ, Γ)-modules for $K \neq \mathbf{Q}_p$ they can also be defined and it is a result of Nakamura that there is a bijection between rank 1 (φ, Γ)-modules and continuous characters $\delta : K^\times \to E^\times$. Finally, the Ext^1 groups were computed by Colmez (in most cases, and by Liu in the remaining cases); they are E-vector spaces of dimension 1 or 2, and in the latter case, the set of extensions is parameterized by a generalization of the $\mathcal{L}$-invariant.

We say that a p-adic representation is potentially trianguline if there exists a finite extension $K/\mathbf{Q}_p$ such that $V|_{G_K}$ is trianguline. Examples of such objects are given by de Rham representations and induced representations. Conversely, we have the following result : if V is a 2-dimensional potentially trianguline representation of $G_{\mathbf{Q}_p}$ then either (1) V is split trianguline, or (2) V is a direct sum of characters or an induced representation or (3) V is a twist of a de Rham representation (these three cases are of course not mutually exclusive). The proof of this result relies on the use of Galois descent : if a triangulation of the (φ, Γ)-module associated to such a representation does not descend, this imposes many conditions on the possible slopes and weights of the occuring rank 1 (φ, Γ)-modules, implying conditions either (2) or (3) (by using Fontaine's theory of $\mathbf{B}_{\mathrm{dR}}$-representations in the latter case).

It is an open problem to find an explicit example of a p-adic representation which is not potentially trianguline, although in recent joint work with Chenevier we show that they do exist.

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