MULTIVARIABLE $(\varphi, \Gamma)$-MODULES FOR THE LUBIN-TATE EXTENSION

by

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1. Introduction

The goal of my talk was to explain some recent progress concerning $(\varphi, \Gamma)$-modules in the “Lubin-Tate” setting. This work was motivated by the $p$-adic local Langlands correspondence for $GL_2(Q_p)$. This correspondence is a bijection between the set of irreducible 2-dimensional $p$-adic representations of $Gal(\overline{Q}_p/Q_p)$ and the set of some $Q_p$-Banach representations of $GL_2(Q_p)$.

The construction of the $p$-adic local Langlands correspondence for $GL_2(Q_p)$ (see for instance [Bre10], [Col10] and [Ber11]) uses the theory of (cyclotomic) $(\varphi, \Gamma)$-modules in an essential way. Consider the ring $Z_p[X]$, and endow it with a Frobenius map $\varphi$ given by $(\varphi f)(X) = f((1+X)^p - 1)$ and with an action of the group $\Gamma = Gal(Q_p(\mu_{p^\infty})/Q_p) \simeq Z_p^\times$ given by $([a] f)(X) = f((1+X)^a - 1)$ if $a \in Z_p^\times$. A (cyclotomic) $(\varphi, \Gamma)$-module is a module $D$ over a ring which contains $Z_p[X]$, and endowed with a semilinear Frobenius map $\varphi$ and a compatible semilinear action of $\Gamma$.

Colmez makes a similar definition and then extends the action of $(Z_p \setminus \{0\} Z_p)$ on $D$ to an action of $GL_2(Q_p)$ on a bigger space.

We can package this data into an action of the monoid $(Z_p \setminus \{0\} Z_p)$ on $D$, with $\varphi$ given by $(p \, 1)$, $[a]$ given by $(a \, 1)$, and multiplication by $(1 + X)^b$ given by $(1 \, b)$. Colmez makes a similar definition and then extends the action of $(Z_p \setminus \{0\} Z_p)$ on $D$ to an action of $GL_2(Q_p)$ on a bigger space.

If we are interested in a $p$-adic local Langlands correspondence for $GL_2(F)$, with $F$ a finite extension of $Q_p$, then the above construction shows that it is possible that we will need $(\varphi, \Gamma)$-modules with $\Gamma \simeq \mathcal{O}_F^\times$, so that instead of working with the cyclotomic extension, we should work with Lubin-Tate extensions.
2. Fontaine’s $(\varphi, \Gamma)$-modules

Let $F$ be a finite Galois extension of $\mathbb{Q}_p$ of degree $h$, let $\pi_F$ be a uniformizer of $\mathcal{O}_F$, let $q$ be the cardinality of $k_F$, let $G_F = \text{Gal}(\overline{\mathbb{Q}_p}/F)$, and let $E = \text{Emb}(F, \overline{\mathbb{Q}_p})$ be the set of embeddings of $F$ into $\overline{\mathbb{Q}_p}$. Let LT be the Lubin-Tate formal group attached to $\pi_F$ and choose some variable $T$ for the formal group law. We then have for every $a \in \mathcal{O}_F$ a power series $[a](T) = a \cdot T + \text{deg} \geq 2$ giving the multiplication-by-$a$ map. Let $\chi_F : G_F \to \mathcal{O}_F^\times$ be the Lubin-Tate character and let $H_F = \ker \chi_F$ and $\Gamma_F = G_F/H_F$. If $F = \mathbb{Q}_p$ and $\pi_F = p$, all of this is the usual cyclotomic data.

Let $Y$ be a variable and let $\mathcal{O}_Y(Y)$ be the set of power series $f(Y) = \sum_{i \in \mathbb{Z}} a_i Y^i$ such that $a_i \in \mathcal{O}_F$ for $i \in \mathbb{Z}$ and $a_i \to 0$ as $i \to -\infty$. Let $\mathcal{E}(Y) = \mathcal{O}_Y(Y)[1/\pi_F]$. This is a two-dimensional local field. We endow it with a relative Frobenius map $\varphi_q$ by $(\varphi_q f)(Y) = f([\pi_F](Y))$, and an action of $\Gamma_F$ by $(gf)(Y) = f([\chi_F(g)](Y))$.

A $(\varphi, \Gamma)$-module over $\mathcal{E}(Y)$ is a finite dimensional $\mathcal{E}(Y)$-vector space endowed with a compatible Frobenius map $\varphi_q$ and a compatible action of $\Gamma_F$. We say that it is étale if it admits a basis in which $\text{Mat}(\varphi_q) \in \text{GL}_d(\mathcal{O}_Y(Y))$. By a theorem of Kisin-Ren (theorem 1.6 of [KR09]), based on the constructions [Fon90] of Fontaine, there is an equivalence of categories between $\{F\text{-linear representations of } G_F\}$ and $\{\text{étale } (\varphi, \Gamma)\text{-modules over } \mathcal{E}(Y)\}$. Let $D(V)$ denote the étale $(\varphi, \Gamma)$-module over $\mathcal{E}(Y)$ attached to a representation $V$.

The $(\varphi, \Gamma)$-module $D(V)$ is useful if one can relate it to $p$-adic Hodge theory, in particular the ring $\mathcal{B}_{\text{dR}}$ and its subrings [Fon94]. This is possible if $D(V)$ is overconvergent, that is if it admits a basis in which $\text{Mat}(\varphi_q)$ and $\text{Mat}(g)$, for $g \in \Gamma_F$, belong to $\text{GL}_d(\mathcal{E}^!(Y))$, where $\mathcal{E}^!(Y)$ denotes the subfield of $\mathcal{E}(Y)$ consisting of those power series $f(Y)$ that have a nonempty domain of convergence. We say that $V$ is overconvergent if $D(V)$ is.

Which representations are overconvergent? If $F = \mathbb{Q}_p$, then all of them are by a theorem of Cherbonnier and Colmez [CC98]. If $F \neq \mathbb{Q}_p$, then not all representations are overconvergent [FX13]. Let us say that an $F$-linear representation $V$ of $G_F$ is $F$-analytic if for all $\tau \in E \setminus \{\text{Id}\}$, $V$ is Hodge-Tate with weights 0 “at $\tau$”, or in other words if $C_p \otimes_F V$ is the trivial $C_p$-semilinear representation of $G_F$. For example $F(\chi_F)$ is $F$-analytic but $F(\chi_{\text{cyc}})$ is not if $F \neq \mathbb{Q}_p$. The following result (theorem 4.2 of [Ber13]) shows that most representations of $G_F$ are not overconvergent if $F \neq \mathbb{Q}_p$: if $V$ is absolutely irreducible and overconvergent, then there is a character $\delta : \Gamma_F \to \mathcal{O}_F^\times$ such that $V(\delta)$ is $F$-analytic.

Conversely, we have the following theorem [Ber14]: if $V$ is $F$-analytic, then it is overconvergent. This theorem had been proved for crystalline representations by Kislin and Ren [KR09], and for some reducible representations by Fourquaux and Xie [FX13].
Kisin and Ren had further suggested that in order to have overconvergent \((\varphi, \Gamma)\)-modules for all \(F\)-representations of \(G_F\), we need rings of power series in \([F: \mathbb{Q}_p]\) variables, one for each \(\tau \in \mathbb{E}\). Later on we will see how to achieve this with the variables \(\{Y_\tau\}_{\tau \in \mathbb{E}}\) where 
\[g(Y_\tau) = [\chi_F(g)]^\tau(Y_\tau)\]
if \(f(T) = \sum a_i T^i\) with \(a_i \in \mathcal{O}_F\), then \(f^\tau(T) = \sum \tau(a_i)T^i\).

3. Construction of overconvergent \((\varphi, \Gamma)\)-modules

We start by reviewing overconvergent \((\varphi, \Gamma)\)-modules in the cyclotomic setting. Let \(F = \mathbb{Q}_p\) and \(\pi_F = p\) and let \(X\) denote the variable \(Y\) above. The \textit{Robba ring} \(\mathcal{R}(X)\) is a ring of holomorphic power series, which contains \(\mathcal{E}^{\dagger}(X)\). Let \(\mathcal{B} = \mathcal{B}^{\dagger}_{\text{rig}}\) denote one of Fontaine’s big rings of periods [Ber02]. It contains the element \(\pi = [\varepsilon] - 1\) of \(p\)-adic Hodge theory, for which \(g(\pi) = (1 + \pi)^{X_{\text{cycl}}(g)} - 1\) and \(\varphi(\pi) = (1 + \pi)^p - 1\). There is therefore a \(\varphi\) and \(G_{\mathbb{Q}_p}\) compatible injection \(\mathcal{R}(X) \to \mathcal{B}\), sending \(X\) to \(\pi\).

Let \(\mathcal{D}(V)\) denote the \((\varphi, \Gamma)\)-module \(\mathcal{D}^{\dagger}_{\text{rig}}(V)\) over the Robba ring attached to \(V\), whose existence follows from the Cherbonnier-Colmez theorem (we drop the decorations to lighten the notation). In order to construct it, we first descend from \(\mathcal{O}_p\) to \(\mathbb{Q}_p(\mu_{p^\infty})\) by setting \(\mathcal{D}(V) = (\mathcal{B} \otimes_{\mathbb{Q}_p} V)^{H_{\mathbb{Q}_p}}\). There then exists some analogues of Tate’s normalized trace maps [Tat67], \(T_n : \mathcal{D}(V) \to \varphi^{-n}(\mathcal{R}(\pi)) \otimes_{\mathcal{R}} \mathcal{D}(V)\), which allow us to “decomplete” \(\mathcal{D}(V)\). This procedure is analogous to the construction of \(\mathcal{D}_{\text{Sen}}(V)\) in Sen theory [Sen81], where one decompletes \((\mathcal{C}_p \otimes_{\mathbb{Q}_p} V)^{H_{\mathbb{Q}_p}}\) using Tate’s normalized trace maps. This procedure, descent and decompletion, is how the Cherbonnier-Colmez theorem is proved.

The main idea for our construction of multivariable \((\varphi, \Gamma)\)-modules is that there is a different way of decompleting, which is still available in the cases when Tate’s normalized trace maps no longer exist (which is the case as soon as \(F \neq \mathbb{Q}_p\)). If \(W\) is an LF space (i.e., an inductive limit of Fréchet spaces), that is endowed with a continuous action of a \(p\)-adic Lie group \(G\), then following [ST03], we can consider the \textit{locally analytic vectors} of \(W\). We let \(W^{\text{la}}\) be the set of vectors of \(W\) such that the orbit map \(g : W \to g(W)\) is locally analytic on \(G\).

Let \(\mathcal{B}_{\mathbb{Q}_p} = \mathcal{B}^{H_{\mathbb{Q}_p}}\). This is an LF space, with an action of \(\Gamma_{\mathbb{Q}_p} \simeq \mathbb{Z}_p^\times\). We have [Ber14] \((\mathcal{B}_{\mathbb{Q}_p})^{\text{la}} = \bigcup_{n \geq 0} \varphi^{-n}(\mathcal{R}(\pi))\) and \(\mathcal{D}(V)^{\text{la}} = \bigcup_{n \geq 0} \varphi^{-n}(\mathcal{R}(\pi)) \otimes_{\mathcal{R}} \mathcal{D}(V)\). This gives a powerful alternate way of decompleting \(\mathcal{D}(V)\).

If \(F \neq \mathbb{Q}_p\), we proceed in a similar way. Let \(F_0 = F \cap \mathbb{Q}_p^{\text{nr}}\), let \(\mathcal{B} = F \otimes_{F_0} \mathcal{B}^{\dagger}_{\text{rig}}\) and let \(\mathcal{D}(V) = (\mathcal{B} \otimes_F V)^{H_F}\). Using almost étale descent, it is easy to show that \(\mathcal{D}(V)\) is a free \(\mathcal{B}_F\)-module of rank \(d\), stable under \(\varphi_q\) and \(\Gamma_F\). We then have the following theorem [Ber14]: \(\mathcal{D}(V)^{\text{la}}\) is a free \(\mathcal{B}_F^{\text{rig}}\)-module of rank \(d\). It is therefore a \((\varphi, \Gamma)\)-module over \(\mathcal{B}_F^{\text{rig}}\).
4. The structure of $\tilde{B}_F^{la}$

The above theorem is meaningful if we understand the structure of $\tilde{B}_F^{la}$. Using the theory of $p$-adic periods, we can construct [Col02] for each $\tau \in F$ an element $y_\tau \in \tilde{B}_F$ such that $g(y_\tau) = [\chi_F(g)]^\tau(y_\tau)$ if $g \in \Gamma_F$ and $\varphi_q(y_\tau) = [\pi_F]^\tau(y_\tau)$. This way, we get a $(\varphi, \Gamma)$-equivariant map from the Robba ring $\mathcal{R}(\{Y_\tau\}_{\tau \in E})$ in the $h$ variables alluded to at the end of §2 to $\tilde{B}_F$, by sending $Y_\tau$ to $y_\tau$. This map is injective. In addition, it extends to a map $\cup_{n \geq 0} \varphi_q^{-n}(\mathcal{R}(\{Y_\tau\}_{\tau \in E})) \to \tilde{B}_F$, whose image is then dense in $\tilde{B}_F$ for the locally analytic topology [Ber14]. This is why we call $(\varphi, \Gamma)$-modules over $\tilde{B}_F^{la}$ multivariable $(\varphi, \Gamma)$-modules.

We can ask whether $\tilde{D}(V)^{la}$ descends to a nice subring of $\tilde{B}_F^{la}$. The main result of [Ber13] shows that if $V$ is crystalline, then $\tilde{D}(V)^{la}$ descends to a reflexive coadmissible module over the ring $\mathcal{R}^+(\{Y_\tau\}_{\tau \in E})$ of power series that converge on the open unit polydisk.

In general, since the action of $\Gamma_F$ on $\tilde{D}(V)^{la}$ is locally analytic, it extends to an action of $\text{Lie}(\Gamma_F)$. For each $\tau \in E$, there is an element $\nabla_\tau \in F \otimes \text{Lie}(\Gamma_F)$ that is the “derivative in the direction of $\tau$”. Let $t_\tau = \log_{LT}(y_\tau)$, so that $g(t_\tau) = \chi_F(g) \cdot t_\tau$. If $f((Y_\sigma)_\sigma) \in \mathcal{R}(\{Y_\tau\}_{\tau \in E})$, then we have $\nabla_\tau f((y_\sigma)_\sigma) = t_\tau \cdot v_\tau \cdot \partial f((y_\sigma)_\sigma)/\partial Y_\tau$, where $v_\tau$ is a unit. Using these operators, we can prove the theorem to the effect that $F$-analytic representations are overconvergent. First, we can relate Sen theory and $(\varphi, \Gamma)$-modules as in the cyclotomic case [Ber02], and we get [Ber14] that $V$ is Hodge-Tate with weights $0$ at $\tau$ if and only if $\nabla_\tau(\tilde{D}(V)^{la}) \subset t_\tau \cdot \tilde{D}(V)^{la}$. If this is the case, and if $\partial_\tau = t_\tau^{-1} \nabla_\tau$, then $\partial_{\tau}(\tilde{D}(V)^{la}) \subset \tilde{D}(V)^{la}$, so that if $V$ is $F$-analytic, then $\tilde{D}(V)^{la}$ is endowed with a system $\{\partial_\tau\}_{\tau \in E \setminus \{\text{id}\}}$ of $p$-adic partial differential operators, as well as a compatible Frobenius map $\varphi_q$. A monodromy theorem [Ber14] then allows us to show that $(\tilde{D}(V)^{la})^{\partial_{\tau}=0}_{\tau \in E \setminus \{\text{id}\}}$ is free of rank $d$ over $(\tilde{B}_F^{la})^{\partial_{\tau}=0}_{\tau \in E \setminus \{\text{id}\}}$. Finally, we show that $(\tilde{B}_F^{la})^{\partial_{\tau}=0}_{\tau \in E \setminus \{\text{id}\}} = \cup_{n \geq 0} \varphi_q^{-n}(\mathcal{R}(y_{\text{id}}))$. This way, we can descend $\tilde{D}(V)^{la}$ to $\mathcal{R}(Y)$ and then finally prove our theorem, using Kedlaya’s theory of Frobenius slopes [Ked05].

References


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