# MULTIVARIABLE $(\varphi,\Gamma)$-MODULES FOR THE LUBIN-TATE EXTENSION

*by*

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## 1. Introduction

The goal of my talk was to explain some recent progress concerning $(\varphi,\Gamma)$-modules in the "Lubin-Tate" setting. This work was motivated by the $p$-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$. This correspondence is a bijection between the set of irreducible 2-dimensional $p$-adic representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and the set of some $\mathbf{Q}_p$-Banach representations of $\mathrm{GL}_2(\mathbf{Q}_p)$.

The construction of the $p$-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$ (see for instance [**Bre10**], [**Col10**] and [**Ber11**]) uses the theory of (cyclotomic) $(\varphi,\Gamma)$-modules in an essential way. Consider the ring $\mathbf{Z}_p[\![X]\!]$, and endow it with a Frobenius map $\varphi$ given by $(\varphi f)(X) = f((1+X)^p-1)$ and with an action of the group $\Gamma = \mathrm{Gal}(\mathbf{Q}_p(\mu_{p^\infty})/\mathbf{Q}_p) \simeq \mathbf{Z}_p^\times$ given by $([a]f)(X) = f((1+X)^a-1)$ if $a \in \mathbf{Z}_p^\times$. A *(cyclotomic)* $(\varphi,\Gamma)$*-module* is a module $\mathrm{D}$ over a ring which contains $\mathbf{Z}_p[\![X]\!]$, and endowed with a semilinear Frobenius map $\varphi$ and a compatible semilinear action of $\Gamma$.

We can package this data into an action of the monoid $\begin{pmatrix} \mathbf{Z}_p\setminus\{0\} & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix}$ on $\mathrm{D}$, with $\varphi$ given by $\begin{pmatrix} p & \\ & 1 \end{pmatrix}$, $[a]$ given by $\begin{pmatrix} a & \\ & 1 \end{pmatrix}$, and multiplication by $(1+X)^b$ given by $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$. Colmez makes a similar definition and then extends the action of $\begin{pmatrix} \mathbf{Z}_p\setminus\{0\} & \mathbf{Z}_p \\ 0 & 1 \end{pmatrix}$ on $\mathrm{D}$ to an action of $\mathrm{GL}_2(\mathbf{Q}_p)$ on a bigger space.

If we are interested in a $p$-adic local Langlands correspondence for $\mathrm{GL}_2(F)$, with $F$ a finite extension of $\mathbf{Q}_p$, then the above construction shows that it is possible that we will need $(\varphi,\Gamma)$-modules with $\Gamma \simeq \mathcal{O}_F^\times$, so that instead of working with the cyclotomic extension, we should work with Lubin-Tate extensions.

## 2. Fontaine's $(\varphi,\Gamma)$-modules

Let $F$ be a finite Galois extension of $\mathbf{Q}_p$ of degree $h$, let $\pi_F$ be a uniformizer of $\mathcal{O}_F$, let $q$ be the cardinality of $k_F$, let $G_F = \mathrm{Gal}(\overline{\mathbf{Q}}_p/F)$, and let $\mathrm{E} = \mathrm{Emb}(F, \overline{\mathbf{Q}}_p)$ be the set of embeddings of $F$ into $\overline{\mathbf{Q}}_p$. Let LT be the Lubin-Tate formal group attached to $\pi_F$ and choose some variable $T$ for the formal group law. We then have for every $a \in \mathcal{O}_F$ a power series $[a](T) = a \cdot T + \deg \geq 2$ giving the multiplication-by-$a$ map. Let $\chi_F : G_F \to \mathcal{O}_F^\times$ be the Lubin-Tate character and let $H_F = \ker \chi_F$ and $\Gamma_F = G_F/H_F$. If $F = \mathbf{Q}_p$ and $\pi_F = p$, all of this is the usual cyclotomic data.

Let $Y$ be a variable and let $\mathcal{O}_{\mathcal{E}}(Y)$ be the set of power series $f(Y) = \sum_{i \in \mathbf{Z}} a_i Y^i$ such that $a_i \in \mathcal{O}_F$ for $i \in \mathbf{Z}$ and $a_i \to 0$ as $i \to -\infty$. Let $\mathcal{E}(Y) = \mathcal{O}_{\mathcal{E}}(Y)[1/\pi_F]$. This is a two-dimensional local field. We endow it with a relative Frobenius map $\varphi_q$ by $(\varphi_q f)(Y) = f([\pi_F](Y))$, and an action of $\Gamma_F$ by $(gf)(Y) = f([\chi_F(g)](Y))$.

A $(\varphi,\Gamma)$-*module over* $\mathcal{E}(Y)$ is a finite dimensional $\mathcal{E}(Y)$-vector space endowed with a compatible Frobenius map $\varphi_q$ and a compatible action of $\Gamma_F$. We say that it is *étale* if it admits a basis in which $\mathrm{Mat}(\varphi_q) \in \mathrm{GL}_d(\mathcal{O}_{\mathcal{E}}(Y))$. By a theorem of Kisin-Ren (theorem 1.6 of [**KR09**]), based on the constructions [**Fon90**] of Fontaine, there is an equivalence of categories between $\{F$-linear representations of $G_F\}$ and $\{$étale $(\varphi,\Gamma)$-modules over $\mathcal{E}(Y)\}$. Let $\mathrm{D}(V)$ denote the étale $(\varphi,\Gamma)$-module over $\mathcal{E}(Y)$ attached to a representation $V$.

The $(\varphi,\Gamma)$-module $\mathrm{D}(V)$ is useful if one can relate it to $p$-adic Hodge theory, in particular the ring $\mathbf{B}_{\mathrm{dR}}$ and its subrings [**Fon94**]. This is possible if $\mathrm{D}(V)$ is *overconvergent*, that is if it admits a basis in which $\mathrm{Mat}(\varphi_q)$ and $\mathrm{Mat}(g)$, for $g \in \Gamma_F$, belong to $\mathrm{GL}_d(\mathcal{E}^\dagger(Y))$, where $\mathcal{E}^\dagger(Y)$ denotes the subfield of $\mathcal{E}(Y)$ consisting of those power series $f(Y)$ that have a nonempty domain of convergence. We say that $V$ is overconvergent if $\mathrm{D}(V)$ is.

Which representations are overconvergent? If $F = \mathbf{Q}_p$, then all of them are by a theorem of Cherbonnier and Colmez [**CC98**]. If $F \neq \mathbf{Q}_p$, then not all representations are overconvergent [**FX13**]. Let us say that an $F$-linear representation $V$ of $G_F$ is $F$-*analytic* if for all $\tau \in \mathrm{E} \setminus \{\mathrm{Id}\}$, $V$ is Hodge-Tate with weights 0 "at $\tau$", or in other words if $\mathbf{C}_p \otimes_F^\tau V$ is the trivial $\mathbf{C}_p$-semilinear representation of $G_F$. For example $F(\chi_F)$ is $F$-analytic but $F(\chi_{\mathrm{cyc}})$ is not if $F \neq \mathbf{Q}_p$. The following result (theorem 4.2 of [**Ber13**]) shows that most representations of $G_F$ are not overconvergent if $F \neq \mathbf{Q}_p$: if $V$ is absolutely irreducible and overconvergent, then there is a character $\delta : \Gamma_F \to \mathcal{O}_F^\times$ such that $V(\delta)$ is $F$-analytic.

Conversely, we have the following theorem [**Ber14**]: if $V$ is $F$-analytic, then it is overconvergent. This theorem had been proved for crystalline representations by Kisin and Ren [**KR09**], and for some reducible representations by Fourquaux and Xie [**FX13**].

Kisin and Ren had further suggested that in order to have overconvergent $(\varphi,\Gamma)$-modules for all $F$-representations of $G_F$, we need rings of power series in $[F:\mathbf{Q}_p]$ variables, one for each $\tau \in$ E. Later on we will see how to achieve this with the variables $\{Y_\tau\}_{\tau\in E}$ where $g(Y_\tau) = [\chi_F(g)]^\tau(Y_\tau)$ (if $f(T) = \sum a_i T^i$ with $a_i \in \mathcal{O}_F$, then $f^\tau(T) = \sum \tau(a_i)T^i$).

## 3. Construction of overconvergent $(\varphi,\Gamma)$-modules

We start by reviewing overconvergent $(\varphi,\Gamma)$-modules in the cyclotomic setting. Let $F = \mathbf{Q}_p$ and $\pi_F = p$ and let $X$ denote the variable $Y$ above. The *Robba ring* $\mathcal{R}(X)$ is a ring of holomorphic power series, which contains $\mathcal{E}^\dagger(X)$. Let $\widetilde{\mathbf{B}} = \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ denote one of Fontaine's big rings of periods [**Ber02**]. It contains the element $\pi = [\varepsilon]-1$ of $p$-adic Hodge theory, for which $g(\pi) = (1+\pi)^{\chi_{\mathrm{cyc}}(g)} - 1$ and $\varphi(\pi) = (1+\pi)^p - 1$. There is therefore a $\varphi$-and-$G_{\mathbf{Q}_p}$ compatible injection $\mathcal{R}(X) \to \widetilde{\mathbf{B}}$, sending $X$ to $\pi$.

Let $\mathrm{D}(V)$ denote the $(\varphi,\Gamma)$-module $\mathrm{D}^\dagger_{\mathrm{rig}}(V)$ over the Robba ring attached to $V$, whose existence follows from the Cherbonnier-Colmez theorem (we drop the decorations to lighten the notation). In order to construct it, we first descend from $\overline{\mathbf{Q}}_p$ to $\mathbf{Q}_p(\mu_{p^\infty})$ by setting $\widetilde{\mathrm{D}}(V) = (\widetilde{\mathbf{B}} \otimes_{\mathbf{Q}_p} V)^{H_{\mathbf{Q}_p}}$. There then exists some analogues of Tate's normalized trace maps [**Tat67**], $T_n : \widetilde{\mathrm{D}}(V) \to \varphi^{-n}(\mathcal{R}(\pi)) \otimes_{\mathcal{R}} \mathrm{D}(V)$, which allow us to "decomplete" $\widetilde{\mathrm{D}}(V)$. This procedure is analogous to the construction of $\mathrm{D}_{\mathrm{Sen}}(V)$ in Sen theory [**Sen81**], where one decompletes $(\mathbf{C}_p \otimes_{\mathbf{Q}_p} V)^{H_{\mathbf{Q}_p}}$ using Tate's normalized trace maps. This procedure, descent and decompletion, is how the Cherbonnier-Colmez theorem is proved.

The main idea for our construction of multivariable $(\varphi,\Gamma)$-modules is that there is a different way of decompleting, which is still available in the cases when Tate's normalized trace maps no longer exist (which is the case as soon as $F \neq \mathbf{Q}_p$). If $W$ is an LF space (i.e., an inductive limit of Fréchet spaces), that is endowed with a continuous action of a $p$-adic Lie group $G$, then following [**ST03**], we can consider the *locally analytic vectors* of $W$. We let $W^{\mathrm{la}}$ be the set of vectors of $W$ such that the orbit map $g \mapsto g(w)$ is locally analytic on $G$.

Let $\widetilde{\mathbf{B}}_{\mathbf{Q}_p} = \widetilde{\mathbf{B}}^{H_{\mathbf{Q}_p}}$. This is an LF space, with an action of $\Gamma_{\mathbf{Q}_p} \simeq \mathbf{Z}_p^\times$. We have [**Ber14**] $(\widetilde{\mathbf{B}}_{\mathbf{Q}_p})^{\mathrm{la}} = \cup_{n\geq 0}\varphi^{-n}(\mathcal{R}(\pi))$ and $\widetilde{\mathrm{D}}(V)^{\mathrm{la}} = \cup_{n\geq 0}\varphi^{-n}(\mathcal{R}(\pi)) \otimes_{\mathcal{R}} \mathrm{D}(V)$. This gives a powerful alternate way of decompleting $\widetilde{\mathrm{D}}(V)$.

If $F \neq \mathbf{Q}_p$, we proceed in a similar way. Let $F_0 = F \cap \mathbf{Q}_p^{\mathrm{unr}}$, let $\widetilde{\mathbf{B}} = F \otimes_{F_0} \widetilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ and let $\widetilde{\mathrm{D}}(V) = (\widetilde{\mathbf{B}} \otimes_F V)^{H_F}$. Using almost étale descent, it is easy to show that $\widetilde{\mathrm{D}}(V)$ is a free $\widetilde{\mathbf{B}}_F$-module of rank $d$, stable under $\varphi_q$ and $\Gamma_F$. We then have the following theorem [**Ber14**]: $\widetilde{\mathrm{D}}(V)^{\mathrm{la}}$ is a free $\widetilde{\mathbf{B}}_F^{\mathrm{la}}$-module of rank $d$. It is therefore a $(\varphi,\Gamma)$-module over $\widetilde{\mathbf{B}}_F^{\mathrm{la}}$.

## 4. The structure of $\widetilde{\mathbf{B}}_F^{\mathrm{la}}$

The above theorem is meaningful if we understand the structure of $\widetilde{\mathbf{B}}_F^{\mathrm{la}}$. Using the theory of $p$-adic periods, we can construct [**Col02**] for each $\tau \in \mathrm{E}$ an element $y_\tau \in \widetilde{\mathbf{B}}_F$ such that $g(y_\tau) = [\chi_F(g)]^\tau(y_\tau)$ if $g \in \Gamma_F$ and $\varphi_q(y_\tau) = [\pi_F]^\tau(y_\tau)$. This way, we get a $(\varphi, \Gamma)$-equivariant map from the Robba ring $\mathcal{R}(\{Y_\tau\}_{\tau \in E})$ in the $h$ variables alluded to at the end of §2 to $\widetilde{\mathbf{B}}_F$, by sending $Y_\tau$ to $y_\tau$. This map is injective. In addition, it extends to a map $\cup_{n \geq 0} \varphi_q^{-n}(\mathcal{R}(\{Y_\tau\}_{\tau \in E})) \to \widetilde{\mathbf{B}}_F$, whose image is then dense in $\widetilde{\mathbf{B}}_F$ for the locally analytic topology [**Ber14**]. This is why we call $(\varphi, \Gamma)$-modules over $\widetilde{\mathbf{B}}_F^{\mathrm{la}}$ *multivariable* $(\varphi, \Gamma)$*-modules.*

We can ask whether $\widetilde{\mathrm{D}}(V)^{\mathrm{la}}$ descends to a nice subring of $\widetilde{\mathbf{B}}_F^{\mathrm{la}}$. The main result of [**Ber13**] shows that if $V$ is crystalline, then $\widetilde{\mathrm{D}}(V)^{\mathrm{la}}$ descends to a reflexive coadmissible module over the ring $\mathcal{R}^+(\{Y_\tau\}_{\tau \in E})$ of power series that converge on the open unit polydisk.

In general, since the action of $\Gamma_F$ on $\widetilde{\mathrm{D}}(V)^{\mathrm{la}}$ is locally analytic, it extends to an action of $\mathrm{Lie}(\Gamma_F)$. For each $\tau \in \mathrm{E}$, there is an element $\nabla_\tau \in F \otimes \mathrm{Lie}(\Gamma_F)$ that is the "derivative in the direction of $\tau$". Let $t_\tau = \log_{\mathrm{LT}}^\tau(y_\tau)$, so that $g(t_\tau) = \chi_F^\tau(g) \cdot t_\tau$. If $f((Y_\sigma)_\sigma) \in \mathcal{R}(\{Y_\tau\}_{\tau \in E})$, then we have $\nabla_\tau f((y_\sigma)_\sigma) = t_\tau \cdot v_\tau \cdot \partial f((y_\sigma)_\sigma)/\partial Y_\tau$ where $v_\tau$ is a unit. Using these operators, we can prove the theorem to the effect that $F$-analytic representations are overconvergent. First, we can relate Sen theory and $(\varphi, \Gamma)$-modules as in the cyclotomic case [**Ber02**], and we get [**Ber14**] that $V$ is Hodge-Tate with weights 0 at $\tau$ if and only if $\nabla_\tau(\widetilde{\mathrm{D}}(V)^{\mathrm{la}}) \subset t_\tau \cdot \widetilde{\mathrm{D}}(V)^{\mathrm{la}}$. If this is the case, and if $\partial_\tau = t_\tau^{-1} \nabla_\tau$, then $\partial_\tau(\widetilde{\mathrm{D}}(V)^{\mathrm{la}}) \subset \widetilde{\mathrm{D}}(V)^{\mathrm{la}}$, so that if $V$ is $F$-analytic, then $\widetilde{\mathrm{D}}(V)^{\mathrm{la}}$ is endowed with a system $\{\partial_\tau\}_{\tau \in \mathrm{E} \setminus \{\mathrm{Id}\}}$ of $p$-adic partial differential operators, as well as a compatible Frobenius map $\varphi_q$. A monodromy theorem [**Ber14**] then allows us to show that $(\widetilde{\mathrm{D}}(V)^{\mathrm{la}})^{\partial_\tau = 0 \text{ for } \tau \in \mathrm{E} \setminus \{\mathrm{Id}\}}$ is free of rank $d$ over $(\widetilde{\mathbf{B}}_F^{\mathrm{la}})^{\partial_\tau = 0 \text{ for } \tau \in \mathrm{E} \setminus \{\mathrm{Id}\}}$. Finally, we show that $(\widetilde{\mathbf{B}}_F^{\mathrm{la}})^{\partial_\tau = 0 \text{ for } \tau \in \mathrm{E} \setminus \{\mathrm{Id}\}} = \cup_{n \geq 0} \varphi_q^{-n}(\mathcal{R}(y_{\mathrm{Id}}))$. This way, we can descend $\widetilde{\mathrm{D}}(V)^{\mathrm{la}}$ to $\mathcal{R}(Y)$ and then finally prove our theorem, using Kedlaya's theory of Frobenius slopes [**Ked05**].

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