Formal groups and p-adic dynamical systems

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I started my talk by explaining some results about formal groups that can be proved using ideas coming from Lubin's theory of p-adic dynamical systems. Let K be a finite extension of $\mathbf{Q}_p$, with integers $\mathcal{O}_K$, and let $F(X,Y) \in \mathcal{O}_K[\![X,Y]\!]$ be a formal group law over $\mathcal{O}_K$. Let $\mathrm{Tors}(F)$ denote the set of torsion points of F in $\mathfrak{m}_{\mathbf{C}_p}$. To what extent is F determined by its torsion points? The first result is that if two formal groups F and G have infinitely many torsion points in common, then $F = G$. The proof of this theorem rests on a rigidity result: if F is a formal group and if $h(X) \in X \cdot \mathcal{O}_K[\![X]\!]$ is such that $h(z) \in \mathrm{Tors}(F)$ for infinitely many $z \in \mathrm{Tors}(F)$, then h is an endomorphism of F. When $F = \mathbf{G}_m$, such a rigidity result had already been proved by Hida. The proofs of these theorems rest on (1) power series arguments inspired by Lubin's theory of p-adic dynamical systems and (2) the fact that if F is a formal group of finite height, then the image of the attached Galois representation contains an open subgroup of $\mathbf{Z}_p^\times \cdot \mathrm{Id}$. This fact follows from a theorem of Serre and Sen.

After discussing the proofs of these theorems, I gave a brief survey of some of Lubin's results on p-adic dynamical systems. I introduced the notion of a Lubin pair, namely a pair (f,u) of elements of $X \cdot \mathcal{O}_K[\![X]\!]$ that commute under composition, with f and u stable, and with f noninvertible and u invertible. I discussed Lubin's observation that given a Lubin pair, there must be a formal group somehow in the background. For example, if $K = \mathbf{Q}_p$ and if (f,u) is a Lubin pair in which f and all of its iterates have simple roots, and $f \not\equiv 0 \bmod p$, then f and u are endomorphisms of a formal group over $\mathbf{Z}_p$. In general, I conjectured that given a Lubin pair (f,u) with $f \not\equiv 0 \bmod \mathfrak{m}_K$, there is a formal group S such that f and u are semiconjugate to endomorphisms of S.

I finished by explaining my motivation for considering p-adic dynamical systems. They occur in the study of (φ,Γ)-modules. If K_∞/K is a sufficiently ramified (more precisely: strictly APF) Galois extension, and if $\Gamma = \mathrm{Gal}(K_\infty/K)$, then the field of norms of K_∞/K is a local field of characteristic p, endowed with a Frobenius map φ and an action of Γ. In order to have a theory of (φ,Γ)-modules for this Γ, we need to lift these actions to a ring of characteristic zero, such as $\mathcal{O}_K[\![X]\!]$. Such a lift gives rise to a p-adic dynamical system, and using Lubin's results we can prove that if such a lift exists, then K_∞/K is abelian. A recent result of Léo Poyeton then says that K_∞/K is generated by the torsion points of a relative Lubin-Tate group S, and that the power series that give the lifts of φ and of the elements of Γ are semiconjugate to endomorphisms of S.

References

[Ber14] L. Berger, *Lifting the field of norms*, J. École polytechnique - Math. 1 (2014), 29–38.

[Ber19a] L. Berger, *Nonarchimedean dynamical systems and formal groups*, Proc. Amer. Math. Soc. 147 (2019), no. 4, 1413–1419.

[Ber19b] L. Berger, *Rigidity and unlikely intersection for formal groups*, preprint, 2019.

[Fon90] J.-M. Fontaine, *Représentations p-adiques des corps locaux. I*, in The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, 1990, 249–309.

[Hid14] H. Hida, *Hecke fields of Hilbert modular analytic families*, in Automorphic forms and related geometry: assessing the legacy of I. I. Piatetski-Shapiro, Contemp. Math., vol. 614, Amer. Math. Soc., Providence, RI, 2014, 97–137.

[Lub94] J. Lubin, *Nonarchimedean dynamical systems*, Compositio Math. 94 (1994), no. 3, 321–346.

[Sen73] S. Sen, *Lie algebras of Galois groups arising from Hodge-Tate module*, Ann. of Math. (2) 97 (1973), 160–170.

[Ser67] J.-P. Serre, *Sur les groupes de Galois attachés aux groupes p-divisibles*, in Proc. Conf. Local Fields (Driebergen, 1966), Springer, Berlin, 1967, 118–131.