FINAL

ADVANCED ALGEBRA

Course notes are allowed. Answers can be given in French. Throughout the exam, A is a ring.

1. Examples

For each question below, give an example of some objects satisfying the given properties (and prove that your answer is correct).

1.1. An exact sequence $0 \to K \to M \to N \to 0$ such that M is not isomorphic to $K \oplus N$.

1.2. Two modules M and N such that $M \neq 0$ and $N \neq 0$ but $M \otimes N = 0$.

1.3. A module M such that $M \neq 0$, $M_{\text{tor}} = 0$ and $M^* = 0$.

2. Projective modules

2.1. Prove that if M is a finitely generated and projective A-module, then M is finitely presented.

3. Nonzero tensor products

3.1. Let M and N be two A-modules. Let B be an A-algebra. Prove that there is an isomorphism of B-modules $(B \otimes_A M) \otimes_B (B \otimes_A N) \simeq B \otimes_A (M \otimes_A N)$.

3.2. Assume that A is a local ring and that M and N are finitely generated A-modules. Prove that if $M \otimes_A N = 0$, then either $M = 0$ or $N = 0$.

4. Double localization

Let M be an A-module and let S and T be multiplicative subsets of A, with $S \subset T$. Let T_S be the image of T in $S^{-1}A$ under the map $\phi_S : A \to S^{-1}A$ given by $a \mapsto a/1$.

4.1. Prove that the map $T^{-1}A \to T_S^{-1}(S^{-1}A)$ given by $a/t \mapsto (a/1)/(t/1)$ is an isomorphism of rings.

4.2. Prove that the map $T^{-1}M \to T_S^{-1}(S^{-1}M)$ given by $m/t \mapsto (m/1)/(t/1)$ is then an isomorphism of $T^{-1}A$-modules.

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5. Locally free modules

In this exercise, A is a noetherian ring and M is a finitely generated A-module. The letter $\mathfrak{p}$ denotes a prime ideal of A. Recall that if $f \neq 0$, then $M_f = M[1/f] = S^{-1}M$ where $S = \{f^n,\, n \geq 0\}$ and that $M_\mathfrak{p} = T^{-1}M$ where $T = A \setminus \mathfrak{p}$.

5.1. Assume that there exists $n \geq 1$ and $f_1, \dots, f_n \in A$ such that $(f_1, \dots, f_n) = A$ and such that M_{f_i} is a free A_{f_i}-module for every $1 \leq i \leq n$. Prove that for every prime ideal $\mathfrak{p}$ of A, the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ is free. Prove that M is projective.

We now prove the converse of this result.

5.2. Let Q be a finitely generated A-module. Prove that if $Q_\mathfrak{p} = 0$, then there exists $s \in A \setminus \mathfrak{p}$ such that $Q_s = 0$.

5.3. Let N be a finitely generated A-module, and let $f : M \to N$ be a map such that $f_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is surjective.

Show that there exists $s \in A \setminus \mathfrak{p}$ such that $f_s : M_s \to N_s$ is surjective.

5.4. Prove that if $M_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module, then there exists $s \in A \setminus \mathfrak{p}$ such that M_s is a free A_s-module.

5.5. Prove that if M is projective, then there exists $n \geq 1$ and $f_1, \dots, f_n \in A$ such that $(f_1, \dots, f_n) = A$ and such that M_{f_i} is a free A_{f_i}-module for every $1 \leq i \leq n$ (hint: consider the set E of those $s \in A$ such that M_s is a free A_s-module).

6. A generalization of Gauss' lemma

Let A be a subring of a domain B, such that A is integrally closed in B.

6.1. Let f and g be two monic polynomials with coefficients in B. Prove that if fg has coefficients in A, then f and g have coefficients in A.

Hint: let K be an algebraically closed field that contains $\mathrm{Frac}(B)$ and let $x_1, \dots, x_r$ be the roots of f in K. What can you say about the ring $A[x_1, \dots, x_r]$?

6.2. Let $f(X) \in B[X]$ be integral over $A[X]$, so that we have an equation $P(f) = 0$ with $P(T) = T^m + p_{m-1}T^{m-1} + \cdots + p_1 T + p_0$, where $p_0, \dots, p_{m-1} \in A[X]$.

Take $r \geq 1$ and let $g(X) = f(X) - X^r$. Let $Q(T) = P(T + X^r) \in A[X][T]$ so that $Q(g) = 0$, and write $Q(T) = T^m + q_{m-1}T^{m-1} + \cdots + q_0$.

Prove that if r is large enough, then q_0 and $-g$ are monic polynomials.

6.3. Prove that f belongs to $A[X]$, and therefore that if A is integrally closed in a domain B, then $A[X]$ is integrally closed in $B[X]$. Hint: use question 6.1.