EXAM

ADVANCED ALGEBRA

Course notes allowed. If you are a native French speaker, give your answers in French. In all of the midterm, A is a commutative ring.

1. Automorphisms

1.1. Let P be the $\mathbf{Z}$-module $\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$ and consider the two submodules $M = 2\cdot\mathbf{Z}\times\mathbf{Z}/2\mathbf{Z}$ and $N = \mathbf{Z}\times\{0\}$ of P. Prove that P/M and P/N are isomorphic but that M and N are not isomorphic.

Let A be a PID and let r be an integer ≥ 1. Let M and N be two submodules of A^r.

1.2. Prove that there exists an automorphism f of A^r such that $f(M) = N$ if and only if A^r/M and A^r/N are isomorphic A-modules.

2. Faithfully flat localizations

We say that an A-module P is faithfully flat if and only if for every map $M' \to M$ between two A-modules, $P\otimes_A M' \to P\otimes_A M$ is injective if and only if $M' \to M$ is injective.

2.1. Prove that P is faithfully flat if and only if P is flat, and for every A-module N, $P\otimes_A N = 0$ implies that $N = 0$.

2.2. Let B be an A-algebra, via a ring homomorphism $f : A \to B$, and take $x \in A$ such that $f(x) \in B^\times$. Compute $A/xA\otimes_A B$.

2.3. Let S be a multiplicative subset of A containing 1. Prove that $S^{-1}A$ is faithfully flat over A if and only if $S^{-1}A = A$.

3. Kähler differentials

Let A be a subring of a ring B. Give the A-algebra $B\otimes_A B$ a B-module structure by $b\cdot(s\otimes t) = bs\otimes t$. Multiplication in B gives rise to a ring homomorphism $m : B\otimes_A B \to B$. Let $I = \ker(m)$ and let $\Omega_{B/A} = I/I^2$. If $s \in B$, let ds be the class of $1\otimes s - s\otimes 1$ in $\Omega_{B/A}$.

3.1. Prove that I is a B-module, and that $\{1\otimes s - s\otimes 1\}_{s\in B}$ generates I as a B-module.

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3.2. Prove that $d : B \to \Omega_{B/A}$ is an A-linear map, that $d(st) = s \cdot dt + t \cdot ds$, and that $da = 0$ if $a \in A$.

Let $B = A[X]$ and identify $B \otimes_A B$ with $A[X, Y]$. Let $H = Y - X$.

3.3. Prove that $A[X, Y] = A[X, H]$ and that $I = H \cdot A[X, H]$.

3.4. Prove that $\Omega_{B/A}$ is the free B-module of rank 1 generated by dX. What is the map $d : B \to \Omega_{B/A}$?

We stop assuming that $B = A[X]$. Let M be a B-module. An A-linear derivation from B to M is an A-linear map $d_M : B \to M$ such that $d_M(st) = s \cdot d_M t + t \cdot d_M s$ and $d_M a = 0$ if $a \in A$. Let $\mathrm{Der}_A(B, M)$ denote the set of A-linear derivations from B to M.

3.5. Prove that $h \mapsto h \circ d$ from $\mathrm{Hom}_B(\Omega_{B/A}, M)$ to $\mathrm{Der}_A(B, M)$ is injective.

Recall that if N is an A-module, there is an isomorphism between $\mathrm{Hom}_A(N, M)$ and $\mathrm{Hom}_B(B \otimes_A N, M)$. If $d_M \in \mathrm{Der}_A(B, M) \subset \mathrm{Hom}_A(B, M)$, we can therefore view d_M as an element $h \in \mathrm{Hom}_B(B \otimes_A B, M)$.

3.6. Prove that $h = 0$ on $I^2 \subset B \otimes_A B$, that h gives rise to a map $h \in \mathrm{Hom}_B(\Omega_{B/A}, M)$, and that $d_M = h \circ d$.

3.7. Prove that $h \mapsto h \circ d$ from $\mathrm{Hom}_B(\Omega_{B/A}, M)$ to $\mathrm{Der}_A(B, M)$ is bijective.

4. Noether normalization

Let K be a field and let T, U, V be some variables. Carry out an explicit Noether normalization for the following K-algebras A (i.e. find algebraically independent elements $x_1, \ldots, x_n$ in A such that A is a finitely generated $K[x_1, \ldots, x_n]$-module)

4.1. $A = K[T, T^{-1}]$.

4.2. $A = K[T, U, V]/(TU)$.

5. Kähler differentials II

5.1. (More difficult) Let $B = A[X]/P(X)$. Prove that $\Omega_{B/A} = A[X]/(P(X), P'(X)) \cdot dX$
Hint : use the method of question 3.3.

5.2. Let L/K be a finite separable extension of fields. Show that $\Omega_{L/K} = 0$.

5.3. Let $K = \mathbf{F}_p(T)$ and $L = K(X)$ with $X^p = T$. Compute $\Omega_{L/K}$.