EXAM

LOCAL FIELDS

Course notes allowed. If you are a native French speaker, give your answers in French. Let K be a finite extension of $\mathbf{Q}_p$. If F is a formal group over $\mathcal{O}_K$, then $\mathrm{Tors}(F) = \{z \in \mathfrak{m}_{\mathbf{C}_p}$ such that there exists $n \geq 0$ with $[p^n](z) = 0\}$, and $\log_F$ denotes the logarithm of F.

1. The zeroes of the logarithm

Let F be a formal group over $\mathcal{O}_K$. Recall that H_K denotes the set of $f(X) \in K[\![X]\!]$ whose radius of convergence is ≥ 1.

1.1. Prove that $\log_F(X) \in \mathrm{H}_K$.

1.2. Take $z \in \mathfrak{m}_{\mathbf{C}_p}$. Prove that if $z \neq 0$, then $|[p](z)|_p < |z|_p$.

1.3. Take $z \in \mathfrak{m}_{\mathbf{C}_p}$ such that $\log_F(z) = 0$. What can you say about $\log_F([p](z))$? Prove that the set of zeroes of $\log_F$ is precisely $\mathrm{Tors}(F)$.

2. Torsion of some formal groups

We use the notation and results of exercise 1.

2.1. Take $\alpha \in \mathcal{O}_K$. Prove that $F_\alpha(X, Y) = X + Y + \alpha XY$ is a formal group.
Hint: compute $1 + \alpha F_\alpha$.

2.2. Compute the height of F_α and compute $\mathrm{Tors}(F_\alpha)$.

2.3. Assume that F is a formal group over $\mathcal{O}_K$ of infinite height, namely that $\overline{[p](X)} = 0$ in $k_K[\![X]\!]$. Prove that $\mathrm{Tors}(F)$ is finite.

2.4. Prove that if $\mathrm{Tors}(F) = \{0\}$, then $\log_F(X) \in \mathcal{O}_K[\![X]\!]$ and that F is then isomorphic over $\mathcal{O}_K$ to the additive formal group.

2.5. Prove that if $K/\mathbf{Q}_p$ is unramified, and F is of infinite height, then $\mathrm{Tors}(F) = \{0\}$.

3. Endomorphisms of formal groups

Let F be a formal group of finite height h defined over $\mathcal{O}_K$. Let L be a finite extension of K. Let $\mathrm{End}_L(F) = \{h(X) \in X \cdot \mathcal{O}_L[\![X]\!]$ such that $h(F(X, Y)) = F(h(X), h(Y))\}$. Let $c : \mathrm{End}_L(F) \to \mathcal{O}_L$ be the map given by $c(h) = h'(0)$. In this exercise, it may be convenient to use the power series $\exp_F(X) = \log_F^{\circ -1}(X) \in K[\![X]\!]$.

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3.1. Prove that if $h(X) \in X \cdot \mathcal{O}_L[\![X]\!]$, then $h \in \mathrm{End}_L(F)$ if and only if $\log_F \circ h = h'(0) \cdot \log_F$.

3.2. Prove that there exist polynomials $\{Q_n(T)\}_{n\geq 1}$ in $K[T]$ having the property: if $a \in L$ and $h(X) \in X \cdot L[\![X]\!]$ is such that $\log_F \circ h = a \cdot \log_F$, then $h(X) = \sum_{n\geq 1} Q_n(a)X^n$.

3.3. Prove that c is an injective ring homomorphism, whose image contains $\mathbf{Z}_p$.

3.4. Prove that if $z \in \mathrm{Tors}(F)$ and $h \in \mathrm{End}_L(F)$, then $h(z) \in \mathrm{Tors}(F)$, and that h gives rise to a $\mathbf{Z}_p$-linear map $T_p h : T_p F \to T_p F$.

3.5. Prove that the map $\mathrm{End}_L(F) \to \mathrm{End}(T_p F)$ given by $h \mapsto T_p h$ is an injective homomorphism of $\mathbf{Z}_p$-algebras.

3.6. Let E be the fraction field of $c(\mathrm{End}_L(F))$, so that E is an extension of $\mathbf{Q}_p$ contained in L. Prove that E injects into $\mathrm{M}_h(\mathbf{Q}_p)$, and therefore that $[E : \mathbf{Q}_p] \leq h$.

4. Ramification subgroups

Let L/K be a finite Galois extension, let $G = \mathrm{Gal}(L/K)$, let π be a uniformizer of L, and let $e = e(L/\mathbf{Q}_p)$. Take $i \geq 1$ and $g \in G_i$. Write $g(\pi) = \pi(1 + a)$ with $a \in \mathfrak{m}_L^i$.

4.1. Prove that $(g - 1)(x) \equiv kax \bmod \mathfrak{m}_L^{i+k+1}$ if $x \in \mathfrak{m}_L^k$.

4.2. Prove that if $i > e/(p - 1)$, then $(g^p - 1)(x) \equiv pkax \bmod \mathfrak{m}_L^{i+k+e+1}$ if $x \in \mathfrak{m}_L^k$
Hint: let $s = g - 1$ and write $g^p - 1 = (1 + s)^p - 1$.

4.3. Prove that if $i > e/(p - 1)$ and $g \in G_i \setminus G_{i+1}$, then $g^p \in G_{i+e} \setminus G_{i+e+1}$.

4.4. Use this to show that $G_i = \{1\}$ if $i > e/(p - 1)$.

5. Artin-Schreier extensions

Let K be a finite extension of $\mathbf{Q}_p$, let $e = e(K/\mathbf{Q}_p)$, and take $n \geq 1$ such that $p \nmid n$ and $n < pe/(p - 1)$. Let y be an element of K with $\mathrm{val}_K(y) = -n$.

Let x be a root of $X^p - X = y$ and let $L = K(x)$.

5.1. Prove that the other roots of $X^p - X = y$ are of the form $x + z_i$ with $z_i \in \mathcal{O}_L$ and $z_i \equiv i \bmod \mathfrak{m}_L$. Prove that L/K is Galois of degree p.

5.2. Let $G = \mathrm{Gal}(L/K)$. Prove that $G_n = G$ and $G_{n+1} = \{1\}$.