MIDTERM

ADVANCED ALGEBRA

Course notes are allowed. Answers can be given in French.

1. Surjective endomorphisms

1.1. Let M be a finitely generated A-module and let $f : M \to M$ be a surjective map. Prove that f is a bijection.

2. Tensor products and projective modules

Let P and Q be two A-modules such that $P \otimes Q \simeq A^n$ for some $n \geq 1$. Let L be a free module such that there is an exact sequence $0 \to K \to L \to P \to 0$.

2.1. Show that $A^n \oplus \mathrm{im}(K \otimes Q \to L \otimes Q) \simeq L \otimes Q$.

2.2. Show that P and Q are projective.

2.3. Prove that P and Q are finitely generated. Hint : let $\{p_i \otimes q_i\}_{1 \leq i \leq m}$ be a finite family of simple tensors that generate $P \otimes Q$. Let $\{e_i\}_{1 \leq i \leq m}$ be a basis of A^m, and consider the map $f : A^m \to P$ that sends e_i to p_i. What can you say about $\mathrm{coker}(f) \otimes Q$?

2.4. Find an example of a ring A and of two A-modules P and Q such that $P \otimes Q$ is nonzero projective, but P is not projective.

3. Divisible modules

We say that an A-module M is divisible if for every $a \in A \setminus \{0\}$, the map $\mu_a : M \to M$ given by $m \mapsto am$ is surjective. For example, both $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$ are divisible $\mathbf{Z}$-modules.

3.1. Show that if there exists a nonzero A-module M that is both divisible and free, then A is a field.

3.2. Show that if A is a domain and if there exists a nonzero A-module M that is both divisible and projective, then A is a field.

3.3. Show that if there exists a nonzero A-module M that is both divisible and finitely generated, then A is a field.

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