MIDTERM

ADVANCED ALGEBRA

Course notes allowed. If you are a native French speaker, give your answers in French. In all of the midterm, A is a commutative ring.

1. A criterion for isomorphism

Let $f : X \to Y$ be a map of A-modules such that $f^* : \mathrm{Hom}(Y, P) \to \mathrm{Hom}(X, P)$ is an isomorphism for all A-modules P.

1.1. Prove that f is injective.

1.2. Prove that f is surjective (for example, you could prove that $\mathrm{coker}(f) = 0$).

2. Surjective endomorphisms

In this exercise, M is a nonzero finitely generated A-module. If $a \in A$, let $\mu_a : M \to M$ denote the map $m \mapsto am$.

2.1. Let $f : M \to M$ be a surjective map. Prove that there is a polynomial P in $A[X]$ such that $P(0) = 1$ and $P(f) = 0$ (hint : give M the structure of an $A[X]$-module using the map f. Observe that $M = X \cdot M$). Prove that f is a bijection.

2.2. Assume that $\mu_a : M \to M$ is surjective for all $a \in A \setminus \{0\}$. Prove that A is a field.

3. Divisibility of invariant factors

Let A be a PID. If $P \in \mathrm{M}_r(A)$, let $d_1(P), \dots, d_r(P)$ denote its invariant factors.

3.1. Fix a basis of A^r and let M and N be submodules of rank r of A^r. Let P and R be the matrices of bases of M and N in the fixed basis of A^r. Prove that $N \subset M$ if and only if R is of the form PQ with $Q \in \mathrm{M}_r(A)$.

3.2. Take $d \in A \setminus \{0\}$ and let p be a prime element of A. Prove that $p^k(A/dA) \simeq A/(d/p^\ell)A$ where ℓ is the largest integer $\leq k$ such that p^ℓ divides d.

3.3. Prove that $p^k(A/dA)/p^{k+1}(A/dA)$ is an A/pA-vector space of dimension 1 if p^{k+1} divides d and dimension 0 otherwise.

Date: November 7.

3.4. Suppose that P and Q are two elements of $\mathrm{M}_r(A)$ with $\det(PQ) \neq 0$. Show that $d_i(P)$ divides $d_i(PQ)$ for all i.

4. Tensor products and projective modules

Let P and Q be two A-modules such that $P \otimes Q \simeq A^n$ for some $n \geq 1$. Let L be a free module such that there is an exact sequence $0 \to K \to L \to P \to 0$.

4.1. Show that $A^n \oplus \mathrm{im}(K \otimes Q \to L \otimes Q) \simeq L \otimes Q$.

4.2. Show that P and Q are projective.