MIDTERM

LOCAL FIELDS

Course notes are allowed. If you are a native French speaker, give your answers in French.

1. Tame extensions

Let F be a finite extension of $\mathbf{Q}_p$, and let L/F be a totally ramified extension of degree $d = np^k$ with n prime to p. Let π_L be a uniformizer of L.

1.1. Prove that there exists $u \in 1 + \mathfrak{m}_L$ such that $\pi_L^d \cdot u^{-1}$ is a uniformizer of F.

1.2. Prove that there exists $w \in 1 + \mathfrak{m}_L$ such that $u = w^n$.

1.3. Prove that there exists a subextension $F \subset K \subset L$ such that $[K : F] = n$, and that K then has a uniformizer π_K such that π_K^n is a uniformizer of F.

2. $\mathbf{C}_p$ is not spherically complete

We say that a metric space is spherically complete if for every nested sequence of nonempty balls $B(a_1, r_1) \supset B(a_2, r_2) \supset \cdots$, the intersection $\cap_{n\geq 1} B(a_n, r_n)$ is also nonempty (where $B(a, r) = \{x,\ d(x, a) < r\}$ is the ball of center a and radius r).

2.1. Prove that if $a \in \mathbf{C}_p$ and $r < s$, then $B(a, s)$ contains two disjoint balls of radius r.

2.2. Let $r_n = 1 + 1/n$ and consider a nested sequence $\{B(a_n, r_n)\}$ of balls of $\mathbf{C}_p$. Prove that if the intersection $\cap_{n\geq 1} B(a_n, r_n)$ is nonempty, then it contains an open subset.

2.3. Prove that if $\mathbf{C}_p$ is spherically complete, then $\mathbf{C}_p$ contains an uncountable family of pairwise disjoint open sets.

2.4. Recall that a topological space is called separable if it contains a countable, dense subset. Prove that $\mathbf{C}_p$ is separable (for example, you could construct an inclusion $\overline{\mathbf{Q}} \subset \mathbf{C}_p$ and prove that $\overline{\mathbf{Q}}$ is dense in $\mathbf{C}_p$).

2.5. Prove that $\mathbf{C}_p$ is not spherically complete.

Date: October 26.

3. Zeroes of holomorphic functions

Let K be a finite extension of $\mathbf{Q}_p$ and let k be its residue field. Let π be a uniformizer of $\mathcal{O}_K$. Let $\mathcal{O}_K\{X\}$ denote the set of power series $\sum_{i\geq 0} a_i X^i$ where $a_i \in \mathcal{O}_K$ and $a_i \to 0$.

3.1. Let $q(X) \in k[X]$ be a polynomial with $q(0) \neq 0$ and take $n \geq 1$. Prove that if $a(X) \in k[X]$, then there exist $b(X), c(X) \in k[X]$, with $\deg(b) \leq \deg(q) - 1$, such that $a(X) = X^n b(X) + q(X)c(X)$.

3.2. Take $g(X) = \sum_{i\geq 0} g_i X^i \in \mathcal{O}_K\{X\}$ such that $\max_i |g_i| = 1$. Define n and d by $n = \min\{i,\ |g_i| = 1\}$ and $n+d = \max\{i,\ |g_i| = 1\}$. Let $h_1(X) = g_n + g_{n+1}X + \cdots + g_{n+d}X^d$ and let $u_1(X) = X^n$. Prove that $g(X) = h_1(X)u_1(X) + \pi r_1(X)$ for some $r_1(X) \in \mathcal{O}_K\{X\}$.

3.3. Prove by induction on j that we can write $g(X) = h_j(X)u_j(X) + \pi^j r_j(X)$, where $h_j(X)$ is a polynomial of degree d with $h_j \equiv h_{j-1} \bmod \pi^{j-1}$ and $u_j(X)$ is a polynomial with $u_j \equiv u_{j-1} \bmod \pi^{j-1}$, and $r_j \in \mathcal{O}_K\{X\}$.

3.4. Prove that we can write $g(X) = h(X)u(X)$ where $h(X)$ is a polynomial of degree d whose leading and constant coefficients are in $\mathcal{O}_K^\times$, and $u \in \mathcal{O}_K\{X\}$ with $u(X) \equiv X^n \bmod \pi$.

3.5. Prove that $g(X)$ has d zeroes (counting multiplicities) in $\{z \in \mathcal{O}_{\mathbf{C}_p},\ |z| = 1\}$.

3.6. Take $f(X) = \sum_{i\geq 0} f_i X^i \in \mathrm{H}_K$ and let $r = |\alpha|$ with $\alpha \neq 0 \in \mathfrak{m}_{\mathbf{C}_p}$. Let $d(r) = \max\{i,\ |f_i|r^i = |f|_r\} - \min\{i,\ |f_i|r^i = |f|_r\}$. Prove that $f(X)$ has precisely $d(r)$ zeroes (counting multiplicities) in $\{z \in \mathfrak{m}_{\mathbf{C}_p},\ |z| = r\}$.

3.7. State and prove a "theory of Newton polygons" for elements of H_K.