

# Structure and dynamics of vortices in a non-local model of superfluids

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The aim of this paper is to study the reconnection of vortices in a quantum fluid with a roton minimum, such as superfluid  $^4\text{He}$ . These reconnection events are thought to be at the origin of the emission of thermal excitations and the generation of Kelvin waves. The dynamics of a model of superfluids described by the Gross-Pitaevskii (GP) equations is investigated numerically. The originality of the present approach is the introduction of a non-local two-body interaction potential which yields the roton minimum in the dispersion relation, as evidenced by neutron scattering experiments in  $^4\text{He}$ . We begin with the calibration of the model, namely choosing a peculiar functional form of the interaction potential, under the constraint of being as close as possible to the experimental dispersion curve and, in the same time, preventing spurious local crystallization events that appear naturally in presence of a low-energy roton minimum. We then follow and track the phenomenon of reconnection starting with an initially perpendicular set of two vortices, each of them being independently a fundamental solution of the dynamics. A precise and quantitative study of various quantities characterizing the evolution of this phenomenon is proposed, that includes the evolution of statistics of several dynamical quantities of interest such as density and probability current, the vortex length, and the description in the appropriate Frenet-Serret frame of reference of a observed soliton that propagates along the cores of the vortices. The properties of this soliton are systematically compared to the predictions of the Local Induction Approximation (LIA), showing similarities and differences. The introduction of the roton minimum in the model, at a characteristic length scale of the atomic size, does not change the macroscopic properties of the reconnection event but the microscopic structure of the vortices differs. Structures are generated at the roton scale and helical soliton waves are evidenced along the vortices. However, contrary to what is expected in classical viscous or inviscid incompressible flows, the numerical simulations do not evidence the generation of structures at smaller or larger scales than the typical atomic size.

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## I. INTRODUCTION

Turbulence in quantum fluids is the study of the motions induced by a tangle of quantum vortices, which are created under a large-scale stirring force, or in the presence of a counter-flow generated by a hot source (see for instance the reviews [1–5]). In practice, it can be investigated in a variety of physical systems, e.g. in cold atoms Bose-Einstein condensates [6, 7], superfluid  $^3\text{He}$  [8, 9], or superfluid  $^4\text{He}$  [10, 11]. In this paper, we focus on the case of superfluid  $^4\text{He}$ , which is obtained when liquid  $^4\text{He}$  is cooled at temperatures below  $T_\lambda \approx 2.17\text{K}$  (at saturated vapor pressure). At finite temperature, the fluid is made up of a mixture of two components, one being classical and viscous, governed by the incompressible Navier-Stokes equation, and the other one being inviscid, compressible and potential with localized and quantized singularities (i.e. quantum vortices) hosting the rotational motions. These two components interact in a subtle way through the friction of these vortices onto the viscous component, a phenomenon that allows the decay of fluctuations of the quantum component without thus the action of viscosity. Macroscopically, i.e. at scales larger than the dissipative scale of the classical component, statistics of velocity fluctuations in this mixture look very similar to the ones observed in classical three-dimensional turbulence, as depicted in the phenomenol-

ogy of Kolmogorov [12]. This includes the fine scale structure of turbulence, such as the power-law decrease of the velocity spectrum and higher-order (i.e. intermittent) properties [10, 13, 14], scale-energy transfers (i.e. the skewness phenomenon) [15], and also the global behavior at large scales [16]. Even if some differences have been highlighted between quantum and classical turbulence [17, 18] at the level of an isolated quantum vortex, it is thus tempting to consider that at a finite temperature below  $T_\lambda$ , the two components are locked at each others, implying that quantum vortices self-organize, forming structures (i.e. bundles) such that the overall locally averaged vorticity field of the superfluid component resembles to the one observed in classical turbulence [1–5]. At smaller scales, typically below the mean inter-vortex distance, it is nonetheless expected a decoupling phenomenon between the two components [19], superfluid velocity fluctuations being governed by other phenomena such as Kelvin waves propagation along vortex cores [20, 21].

Such scales are difficult to access experimentally, thus from a modeling point of view, it is tempting to study the collective effects of a population of localized singularities hosting a distributional repartition of vorticity, in particular in interaction with an exterior (classical) velocity field. A popular method to study the interaction between these vortices with a normal component is

given by the Local Induction Approximation (LIA), as initiated in Refs. [22, 23], that takes only into account at the level of a single vortex the induced velocity field of a local portion of the vortex. Such an approach has been generalized to take into account non local effects induced by the whole vortex [24, 25], in order to study the dynamics of an ensemble of vortices and the implication of non local effects. In this context, a phenomenon of tremendous importance is vortex reconnection, that allows for dissipation and a possible change in the macroscopic distribution of these vortices, that has to be put by hand at this stage.

An alternative approach devoted to the dynamics of vortices and their interaction, that might be of some interest in the improvements of the aforementioned discrete approaches, neglecting the coupling with a normal component and assuming vanishing temperature  $T = 0$ , is given by the description and evolution of the order parameter of the superfluid in terms of fields, as it is proposed by a partial differential equation known as the Gross-Pitaevskii (GP) equation [26]. Contrary to the approach based on the LIA, the GP equation includes in an intrinsically fashion vortex reconnection, without thus asking for further modeling steps. This approach has been studied in an extended fashion in the literature [27, 28]. In this approach, that allows to study some global behavior of an assembly of vortices at scales of the order of the inter-vortex distance, their very internal structure is neglected and the two body interaction usually adopted is of localized (i.e. distributional) type. The purpose of this article is the numerical study of a non-local version of the GP equations that allows to reproduce a realistic internal structure of these vortices and observe and quantify its implication on vortex reconnection, and on possible soliton propagation along their cores. This can be done while introducing in the interaction potential a physical length scale  $a$  representing the typical size of a  $^4\text{He}$  atom.

## II. A NON LOCAL MODEL OF SUPERFLUIDS INCLUDING THE ROTON MINIMUM, AND ITS CALIBRATION

At zero temperature  $T = 0$ , to the lowest approximation we can consider the superfluid under study to be described by a scalar wave function  $\psi$ , i.e. an order parameter, which is space and time dependent. Henceforth, we consider dimensionless coordinates  $\mathbf{r} \in \mathbb{R}^3$  and  $t \in \mathbb{R}$  by respectively the roton wavelength  $a = 3.26 \text{ \AA}$  [26, 29] and a quantum typical time  $t_0 = 2ma^2/\hbar = 1.34 \times 10^{-11} \text{ s}$ , where  $m = 6.65 \times 10^{-27} \text{ kg}$  corresponds to the  $^4\text{He}$  atom mass. Considering that the number of atoms is high in the condensate, we can assume that the dynamics of  $\psi$  is given by the GP equation, that reads in its most general and dimensionless version

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + (V * |\psi|^2)\psi - \mu\psi, \quad (1)$$

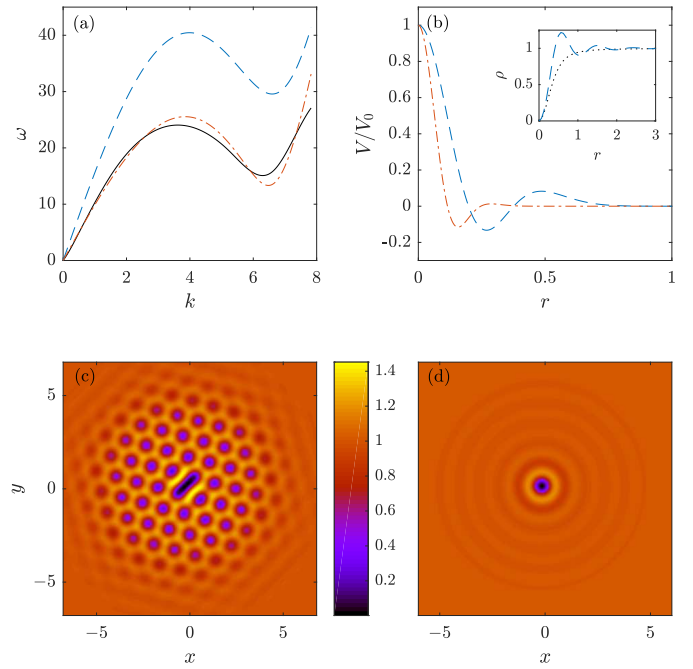


FIG. 1. (a): Dispersion relations of  $^4\text{He}$  and its model as given by the non-local GP equation (Eq. 1): experimental results from neutron scattering experiments (black line, see Ref. [29]), its fit as proposed in Ref. [30] (red dashed-dotted line), our current model (Eq. 3) with a raised roton minimum so as to avoid crystallization as explained in the text (blue dashed line). (b): Corresponding interaction potentials  $V(r)/V_0$  entering in Eq. 1, with  $V_0 = 21.96 \text{ eV}$  for the model of Ref. [30]  $V_0 = 0.15 \text{ eV}$  for our present model (Eq. 3), used to obtain the theoretical dispersion curves of (a). We display in the inset the radial density profile of the stationary solution corresponding to the local GP equation (Eq. 10) with a black dotted line, and the corresponding profile in the non-local case (Eq. 1) with a blue dashed line. (c) and (d) Surface plots of the distribution of density, i.e.  $|\psi|^2$ , of the stationary vortex solution for the potential proposed in Ref. [30], and obtained as the solution of the relaxation problem given in Eq. 4, which leads to crystallization for this model. (d): Same representation of the density distribution as in (c) but with our a raised roton gap (Eq. 3) so as to avoid crystallization.

where  $V(\mathbf{x})$  is a smooth two-body interaction potential, assumed to be spherically symmetrical (a function of the norm  $x = |\mathbf{x}|$  only),  $*$  stands for the convolution product, i.e.  $(V * |\psi|^2)(\mathbf{x}, t) = \int V(\mathbf{x} - \mathbf{y})|\psi|^2(\mathbf{y}, t) d^3y$ , and  $\mu = \int V(\mathbf{x}) d^3x$  the chemical potential ensuring  $|\psi|^2 = 1$  as a homogeneous solution. As it is shown in Refs. [30–32], the finite extension of the interaction potential is crucial in obtaining a dispersion relation that reproduces the roton minimum, as it is observed in neutron scattering measurements performed in superfluid  $^4\text{He}$  [29, 33]. This allows us to calibrate the superfluid model that is proposed in Eq. 1 while computing the dispersion relation, readily obtained as

$$\omega^2(\mathbf{k}) = |\mathbf{k}|^4 + 2|\mathbf{k}|^2\widehat{V}(\mathbf{k}), \quad (2)$$

where  $\widehat{V}(\mathbf{k}) = \int e^{i\mathbf{k}\cdot\mathbf{x}} V(\mathbf{x}) d^3x$  is the Fourier transform of the interaction potential, which depends only on  $k = |\mathbf{k}|$  if  $V$  is taken isotropic. To get the dispersion relation (Eq. 2), we use standard techniques consisting in linearizing Eq. 1 while looking for solution of the form  $\psi = 1 + \varphi$ , for small  $\varphi$ , Fourier transforming the linear dynamics of  $\varphi$  and its conjugate, then looking for constraints on  $\omega$  and  $|\mathbf{k}|$  to avoid a single trivial solution (see [30–32]). Choosing a particular form for  $\widehat{V}(\mathbf{k})$  allows then to compare the so-obtained dispersion relation against experimental measurements.

In subsequent numerical simulations that we will present in the next sections, we choose the following functional isotropic form for the interaction potential

$$\widehat{V}(\mathbf{k}) = \left( \frac{c_s^2}{2} - v_1^2 |\mathbf{k}|^2 + v_2^4 |\mathbf{k}|^4 \right) \exp\left(-\frac{|\mathbf{k}|^2}{2k_0^2}\right), \quad (3)$$

where  $c_s$  corresponds to the sound velocity, i.e. the limit at small wave-vector of  $\omega^2(\mathbf{k})/|\mathbf{k}|$ , and  $(v_1, v_2, k_0)$  three free parameters than could be obtained for example using a least-square fit procedure given a experimental dispersion relation. We will not do that, and use instead, within our choice of units,  $c_s = 16$ ,  $v_1 = 2.2635$ ,  $v_2 = 0.4408$  and  $k_0 = 5.5970$ . We now motivate our choice and compare against experimental data.

We represent in Fig. 1 (a) the dispersion relation of liquid  $^4\text{He}$  at saturated vapor pressure provided in Ref. [29] using a dashed line, and that exhibits indeed a roton minimum around  $|\mathbf{k}| \approx 2 \text{ \AA}^{-1}$ . We superimpose there using a blue dashed line the dispersion relation that we will use in subsequent numerical simulations using Eq. 2 with an interaction potential (Fourier transformed) given in Eq. 3 with the formerly defined free parameters  $(v_1, v_2, k_0)$  and the given sound velocity  $c_s$ . We observe quantitative differences between the experimental and our theoretical dispersion relations. First, when expressed in physical units, we have chosen a sound velocity of the order of 354 m/s, which is higher than the observed value 238 m/s. This explains why experimental and theoretical curves deviate at vanishing  $|\mathbf{k}|$ . Furthermore, we see that the theoretical curve reproduces the correct value of the roton wavevector  $|\mathbf{k}| \approx 2 \text{ \AA}^{-1}$ , but not the value of the of the minimum of angular frequency  $\omega$  (nor consistently the value of the *maxon*, i.e. the local maximum of frequency occurring just before). This choice is dictated by further numerical investigations in which we forbid any crystallization phenomena, a natural tendency of this type of model (Eq. 1) to evolve toward a periodical modulations of density  $\rho = |\psi|^2$ , as it has been exploited to describe a possible supersolid-state of matter [34, 35]. At this stage, let us state that the present approach, based on a scalar wave function with a two-body non-local interaction as considered in Eq. 1 is unable to describe the dynamics of  $^4\text{He}$  in a superfluid phase with a more realistic dispersion relation.

Indeed, let us show that such a choice for the interaction potential (Eq. 3) allows axisymmetric stationary

solutions, i.e. vortex-lines with quantized circulation, as it has been widely studied for the local GP equation [26].

### III. A DETOUR THROUGH THE NUMERICAL ESTIMATION OF AXISYMMETRIC STATIONARY SOLUTIONS

A major success of scalar wave functional approaches, and its related dynamics given by the GP equation [26], lies in the existence of a stationary solution (i.e. time independent) which is axisymmetric (say independent of the  $z$  coordinate and on the polar angle  $\varphi$  in the  $xy$ -plane) and exhibiting a  $2\pi$  defect for the phase. More precisely, a solution of the form  $\psi(r, \varphi, z) = \sqrt{\rho(r)} e^{i\varphi}$ , where we have introduced the polar decomposition of the wave function in terms of amplitude and phase, and the cylindrical coordinates  $(r, \varphi, z)$  in which  $x = r \cos(\varphi)$  and  $y = r \sin(\varphi)$ . To numerically estimate the shape of the density distribution  $\rho(r)$  as a function of the polar distance  $r$ , and more generally to test the existence of such a axisymmetric solution, we numerically solve the two-dimensional relaxation problem (corresponding to the propagation of Eq. 1 in imaginary time with the  $z$ -independence as a constraint)

$$\frac{\partial \psi}{\partial t} = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi - (\tilde{V} * |\psi|^2) \psi + \mu \psi, \quad (4)$$

with initial condition  $\psi(x, y, 0) = e^{i\varphi(x, y)}$ ,  $\varphi(x, y) = \arctan(y/x)$  (the inverse tangent being suitably defined), and  $\tilde{V}(x, y) = \int V(x, y, z) dz$ . We solve this initial value problem using periodic boundary conditions in order to efficiently compute linear operations in the Fourier space, and nonlinear ones in the physical space. Doing so, in order to prevent from phase discontinuities, we use four copies of this initial condition with appropriate phase distribution, and evenly spaced, as it is explained and performed in Ref. [27]. In units of the length scale  $a$ , we use as a mesh size  $dx = 1/16$ , and depending on the number of collocation points  $N$  in each direction, we consider domains of physical size  $Ndx$ . Using thus  $N = 512$  (vortices are sufficiently far apart to neglect their interaction), we simulate a domain of physical size  $32a \approx 100 \text{ \AA}$ . Time propagation is performed using a fourth-order Runge-Kutta explicit method with  $dt = (dx)^2/64$ , corresponding to  $8.2 \times 10^{-16}$  s in physical units. To prevent from spurious generation of unphysical small scales, we use as a dealiasing method the 2/3-rule, each time we perform a multiplication in the physical space. In our case, since the nonlinearity is of order three, we apply this rule three times at each time step, which is enough to prevent the generation of unphysical small scales.

Starting from our initial condition, we observe the convergence of Eq. 4 toward a time-independent solution that we can consider as a stationary solution of the non-local GP equation (Eq. 1) itself, and we represent it in Fig. 1 (d). We see that indeed the solution respects

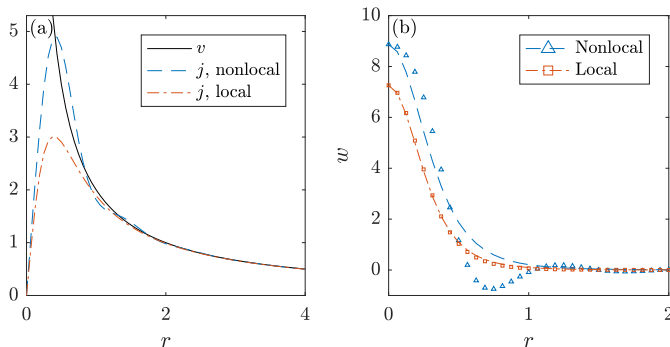


FIG. 2. (a) Radial distributions of the velocity field  $\mathbf{v}$  and probability current  $\mathbf{j}$  of the axisymmetric stationary solutions of the local (Eq. 10) and non-local (Eq. 1) GP equations. (b) Radial distributions of the associated pseudo-vorticity fields (Eq. 8), and their comparison with a schematic fit provided in Eq. 9.

the symmetry with respect to the axis of the vortex, where the density tends to zero, as expected for such a quantum singularity. We furthermore see the existence of additional density oscillation around the vortex core, which are themselves expected once a roton minimum is included in the picture. This is a well-known phenomenon [30, 32, 36–39], for which density oscillation are prescribed by the characteristic shape and size of the roton minimum. We represent in the inset of Fig. 1 (b) a comparison of the density profile away from the location of the singularity obtained in axisymmetric stationary solutions of the local (given in Eq. 10, curve is in black) and non-local GP equations (our present numerical estimation) where we see more clearly that the presence of the roton minimum leads to periodical modulation of density, governed by the roton characteristics, and can be seen as precursors of the crystallization phenomenon.

#### IV. FURTHER COMMENTS ON THE MODEL OF THE INTERACTION POTENTIAL

We represent (red continuous line) in Fig. 1 (a) the model of the non-local interaction potential  $V$  chosen in Ref. [30], which also include an additional quintic term in the overall dynamics. It does reproduce with great accuracy the experimental dispersion relation of Ref. [29]. Nonetheless, when looking for a possible axisymmetric stationary solution, in the spirit of the relaxation problem posed in Eq. 4, we end up eventually with a time-independent solution displayed in Fig. 1 (c). We see that in this case, the invariance around the axis of the singularity is broken by the appearance of additional periodic modulation of density that form a hexagonal structure. This crystallization phenomenon was already observed in early numerical simulations of the non-local GP equation [40], where it is associated to a phenomenon of mass concentration associated to negative values of the interaction potential. Remark also that we could have used

also another set of parameters ( $c_s, v_1, v_2, k_0$ ) in order to be much closer to the experimental dispersion curve, and without including an additional quintic term. Similarly as in the model proposed in Ref. [30], the corresponding stationary vortex solution would also exhibit such a crystallization phenomenon (data not shown).

In the light of more recent studies [34, 35] concerning the natural evolution toward the state of supersolidity when a roton minimum is included in the picture, we are left with concluding that a GP type of evolution, where enters a simple non-local two-body interaction (with a possible additional quintic term), is unable to describe in a proper manner superfluid  $^4\text{He}$  if we follow with great accuracy the experimental dispersion curve of Ref. [29]. Note nonetheless that calibrating our model, which involves a nonlinearity in the evolution (Eq. 1), using the dispersion relation, that is a prediction obtained through a linearization procedure, is difficult to control. It would be of great interest to develop a new type of dynamics that would include both a correct description of the experimental dispersion relation, and the existence of stationary axisymmetric solutions representing in a proper way quantum vortices. We leave this aspect for future investigations, and perform subsequent three dimensional numerical simulations with the aforementioned model for the interaction potential (Eq. 2), that prevents the formation of these hexagonal periodic modulation of density.

Let us now investigate the very radial distribution of various kinematic quantities entering in the hydrodynamical interpretation of the non-local GP equation (Eq. 1), as it is given by the Madelung transformation [26]. In this approach, we associate the gradient of the phase of  $\psi$  to a velocity field  $\mathbf{v}$  and  $|\psi|^2$  to a local density field  $\rho$ . Key kinematic quantities are thus density  $\rho = |\psi|^2$  and probability current  $\mathbf{j} = -i(\psi^* \nabla \psi - \psi \nabla \psi^*) = \rho \mathbf{v}$  that are governed by conservation equations that read [26, 28, 30]

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \quad (5)$$

that can be interpreted as a continuity equation, and considering a component  $j_i$  of the vector  $\mathbf{j}$ , we have

$$\frac{\partial j_i}{\partial t} + \partial_j \Pi_{ij} = 0, \quad (6)$$

where  $\Pi_{ij}$  is the momentum tensor,

$$\Pi_{ij} = \partial_i \psi \partial_j \psi^* - \psi \partial_{ij}^2 \psi^* + \text{c.c.} + \frac{1}{2} \rho (V * \rho) \delta_{ij}, \quad (7)$$

where c.c. stands for the complex conjugate, and  $\delta_{ij}$  the Kronecker symbol. Similarly, we could derive the time evolution of the velocity field  $\mathbf{v}$ , which corresponds to a compressible, irrotational and barotropic fluid with an additional quantum pressure term, of density corresponding to  $|\psi|^2$  (see for instance Refs. [30, 40]). It is well known that the velocity field diverges in the vicinity of a defect of the phase, so we will in the next sections work with the current vector  $\mathbf{j}$ , that is eventually, as we are going to see, a bounded vector.

We display in Fig. 2 (a) the radial profile of velocity  $\mathbf{v}$  and probability current  $\mathbf{j}$  of the axisymmetric stationary solution represented in Fig. 1 (d). Let us recall that, using cylindrical coordinates centered on the vortex, the velocity field  $\mathbf{v} = 1/r \mathbf{e}_\theta$  and the current  $\mathbf{j} = \rho(r)/r \mathbf{e}_\theta$  are known. Remark that  $|\mathbf{v}| = 1/r$ , as we mentioned, diverges in the vicinity of the vortex line, whereas  $\mathbf{j}$  is a bounded vector since  $\rho(r)$  tends to 0 as  $r^2$  [26] in the vicinity of the origin (so that  $\mathbf{j}$  vanishes itself at the origin). This is what is indeed obtained and displayed in Fig. 2 (a). We furthermore superimpose the radial distribution of  $\mathbf{j}$  that is obtained from the local, i.e. standard, version of the GP equation, that we define precisely later (see Eq. 10). Once again, we see that the current follows a non monotonic radial behavior. Compared to what is obtained in the non-local GP equation (Eq. 1), we can see that the maximum of current is obtained at a similar distance from the axis of symmetry in both models, but its value is higher in the non-local version of the dynamics.

Of great interest also from a dynamical point of view is the radial distribution of the pseudo-vorticity

$$\mathbf{w} = \nabla \wedge \mathbf{j}, \quad (8)$$

which quantifies the rotational motions of this compressible fluid. In the following, we note  $w = |\mathbf{w}|$  the amplitude of pseudo-vorticity. For a single vortex line along the  $z$  axis,  $\mathbf{w}$  is aligned with  $\mathbf{e}_z$ , and its amplitude only depends on the radial distance  $r$  to the axis, and we have  $w(r) = (1/r)d\rho(r)/dr$ . Recall that vorticity itself, i.e. the curl of the velocity field  $\mathbf{v}$ , is of distributional nature, and vanishes everywhere except at the very position of the singularity where it diverges. Instead, as we can see in Fig. 2 (b),  $\mathbf{w}$  as defined in Eq. 8, is a bounded quantity.

In the local case (red symbols), we can even see that the radial distribution of pseudo-vorticity, as far as the axisymmetric solution is concerned, follows a monotonic decrease away from its axis of symmetry. As we see in the following, in particular to interpret some quantities entering in a statistical description of the flow, it is interesting to design a model for this radial behavior. For this purpose, we propose the following decreasing function

$$w(r) = \frac{w_0}{\left[1 + \left(\frac{r}{r_0}\right)^2\right]^\alpha}, \quad (9)$$

where  $w_0$  is the value of pseudo-vorticity on the axis of symmetry, and  $(r_0, \alpha)$  two free parameters describing the shape of pseudo-vorticity radial distribution. Starting with the radial distribution of  $\mathbf{w}$  obtained from the local GP equation (and presented later in Eq. 10), the fit (Eq. 9) reproduces in a fairly good way the observed decrease using  $r_0 = 0.47$  (in units of  $a$ ) and  $\alpha = 2.5$ . In a non-local context setting, as given by Eq. 1, such a fit reproduces accurately the decrease with  $r_0 = 0.54$  and same  $\alpha$ , but obviously fails at reproducing the non monotonic behaviors associated to the additional oscillations

associated to the roton minimum. As we will see, even if some aspects are not reproduced by such a schematic fit (Eq. 9), it will be very useful to interpret subsequent statistical quantities that we will observe in the next sections.

## V. NUMERICAL INVESTIGATION OF VORTEX RECONNECTION INITIALLY PERPENDICULAR

Let us now investigate the dynamics of vortex reconnection in the presence of a roton minimum, and thus with a non-local interaction potential (Eq. 1), as it was studied in a local GP equation in Ref. [27]. To do so, we prepare an axisymmetric stationary solution as it is described in the former paragraph, and represented in Fig. 1 (d), properly extended to three dimensions and take as an initial condition the product of two such wave functions, one being in the center of the domain and the other one being shifted from the center by one atomic distance  $a$  and rotated such that we get initially two perpendicular vortices, in a very similar way as in Ref. [27]. One of the purpose of the present article, is to study and quantify the differences in the evolution of this set of two perpendicular vortices with and without the roton minimum. Thus, in addition to three-dimensional numerical simulations of the non-local GP equation (Eq. 1), as we have already discussed in particular in Fig. 2, we perform such a simulation with the standard (local) form of the GP equation. In our system of units, we thus consider also the following dynamics

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + g(|\psi|^2 - 1)\psi, \quad (10)$$

where  $g = 21.345$ . We have chosen this particular value for  $g$  to ensure that the typical vortex core extension (i.e the healing length) is of the same order of what is obtained in the non-local model. In a similar fashion as it is done while considering the relaxation problem of Eq. 4, we solve Eqs. 1 and 10 in a periodic fashion using a pseudo-spectral method, with same  $dx$ ,  $dt$  and dealiasing rule, over  $N^3 = 512^3$  collocation points. Results of both simulations are displayed in Fig. 3, using the visualization software Vapor [41]. Once again, only one eighth of the computational domain is displayed, but we keep in mind that copies remain to warrant a continuous distribution of the phase of the wave function.

We have displayed the evolution of this set of two initially perpendicular vortices at four different time: (i) the initial time  $t = 0$ , (ii) at the time of reconnection  $t = t_{\text{rec}}$ , (iii) after the reconnection at  $t = 2t_{\text{rec}}$  and (iv) some time after the reconnection at  $t = 10t_{\text{rec}}$ . The time of reconnection,  $t_{\text{rec}}$  is 0.79 in the local formulation, and 0.98 in the non-local one, resp.  $1.06 \times 10^{-11}$  s and  $1.31 \times 10^{-11}$  s in physical units. Respective dispersion curves being different, in particular sound velocities, it is not inconsistent to observe two different reconnection

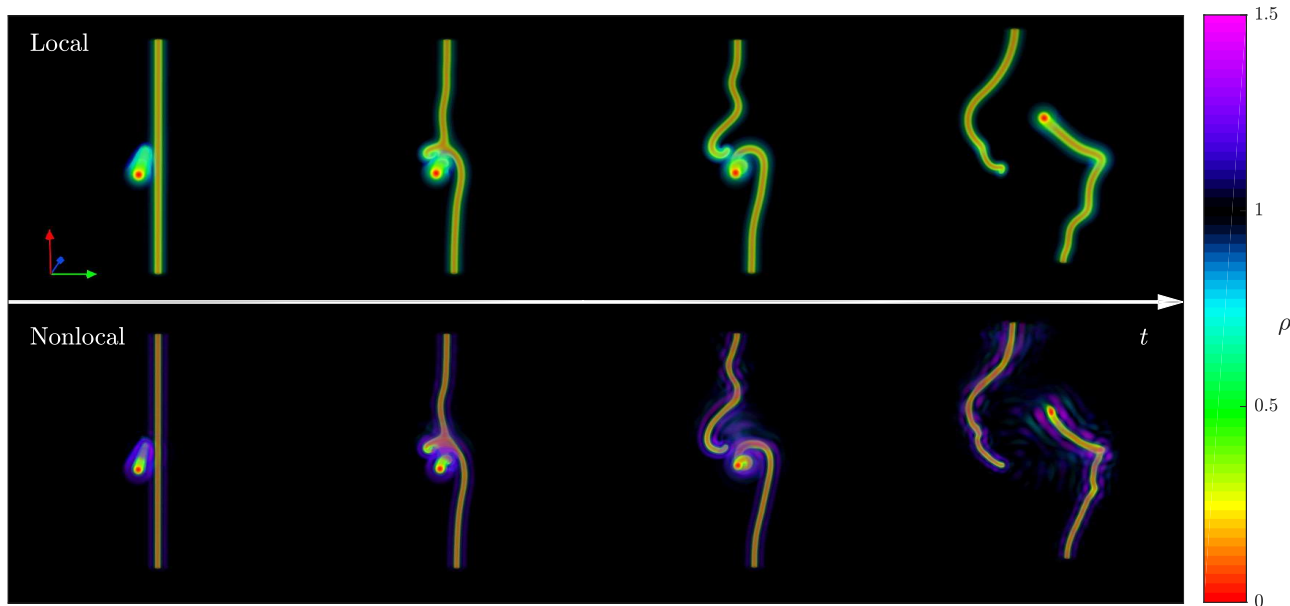


FIG. 3. 3D visualization of the superfluid density during reconnection, in the local (top) and non-local (bottom) models. A density threshold is applied for clarity, so that the bulk fluid density  $\rho \sim 1$  is transparent. Four snapshots are represented along time, from left to right respectively: the initial condition  $t = 0$ , the reconnection time  $t = t_{\text{rec}}$ , twice the reconnection time  $t = 2t_{\text{rec}}$ , and the end time of the simulation  $t = t_{\text{end}} \approx 6t_0$ . It should be noted that because of the different dynamics of the two models,  $t_{\text{rec}}$  takes slightly different values in the local and non-local case (see text).

time  $t_{\text{rec}}$ . However, it is very surprising at this stage that overall the phenomenon of reconnection, from a spatial point of view, looks very similar in the local and non-local formulation. Indeed, the spatial distributions of density  $|\psi|^2$  at the final stage  $t_{\text{end}}$  that we consider in both cases share striking similarities. We can conclude that the internal (i.e. microscopic) structure of the vortices, strongly influenced by the presence of the roton minimum, has nonetheless little influence on the overall global evolution at larger scales than  $a$ . This is far from being obvious since we are solving a nonlinear problem that could have been highly sensitive to the internal density structure of these singularities. From a local point of view, paying attention to the precise values of the densities across time and space, we can see that the non-local GP allows locally high values (of the order of  $\rho \sim 1.5$ ), which we recall do not seem to have implications on the global dynamics.

To quantify more precisely the generation of strong fluctuations of the superfluid density  $\rho = |\psi|^2$ , we compute the probability density function (PDF) of the field of densities obtained in the simulation domain for both the local and non-local cases, at the four times considered in Fig. 3, and we display our results in Figs. 4 (a) and (b). We observe that along the phenomenon of reconnection, densities higher than the uniform one are forbidden in the local case (Eq. 10). On the contrary, in the non-local approach, in which local densities higher than one are initially present due to the roton minimum, the dynamics may develop local mass concentration exceeding

three times the value of the uniform density.

In order to interpret subsequent PDFs that we are going to estimate, let us first focus on the simple axisymmetric stationary solution (i.e. the vortex line) that we presented in Fig. 1 (d). Consider then any physical quantity of interest  $F$ , and its respective PDF  $\mathcal{P}_F$ , i.e. the histogram of the values  $g(x, y, z)$  taken by the quantity  $F$  in the domain  $\mathcal{V}$  (of volume  $|\mathcal{V}|$ ). The PDF can be written as the following empirical average

$$\mathcal{P}_F(f) = \frac{1}{|\mathcal{V}|} \int_{\mathcal{V}} \delta(f - g(x, y, z)) dx dy dz, \quad (11)$$

where  $\delta$  denotes the Dirac function. For a single vortex, in a cylindrical volume  $\mathcal{V}$  of radius  $R$  and of finite height, the PDF of  $\mathbf{v}$  can be computed in an exact fashion, inserting  $g(r, \theta, z) = 1/r$  in the empirical interpretation of the PDF (Eq. 11) and performing a proper change of variable, we get  $\mathcal{P}_{|\mathbf{v}|}(|\mathbf{v}|) = 2R^{-2}|\mathbf{v}|^{-3}$  for  $|\mathbf{v}| \geq R^{-1}$  (and 0 for  $|\mathbf{v}| < R^{-1}$ ), showing that the tail is governed by the divergence of velocity in the vicinity of the vortex, as it was noticed in Ref. [17]. Note that similar power-law behavior  $\sim |\mathbf{v}|^{-3}$  have been observed in simulations of the local GP equation (Eq. 10) as detailed in Ref. [18]. This power-law behavior of the tail of the PDF of the norm of velocity is also observed for the case of two perpendicular vortices, as we consider to initiate our numerical simulations. We have checked in our simulation that this is also the case for the initial condition we are using, for both the local and non-local case, all along the reconnection process (data not shown).

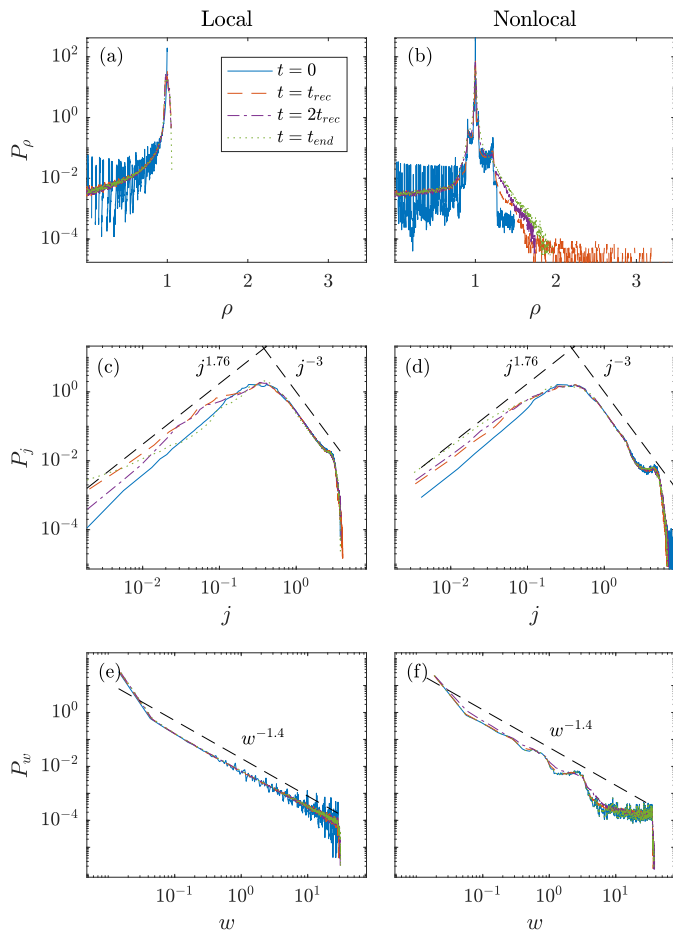


FIG. 4. Plots of the probability density functions (PDF) of various hydrodynamical quantities, in the local (left) and non-local (right) models. (a,b): PDF of the superfluid densities  $|\psi|^2$ . High density events  $\rho > 1$  are well represented in the non-local case (b), whereas they are absent in the local model (a). (c,d): PDF of the probability current norm  $j = |\mathbf{j}|$  (see text for a precise definition). A  $j^{-3}$  behavior is observed on the right tails of both models and comes from the midrange decay of  $j$ . A  $j^{1.76}$  scaling is observable on both left tails. (e,f): PDF of the pseudo-vorticity norm  $w = |\mathbf{w}|$  (Eq. 8). The observed power-law behaviors  $P_w \sim w^{-1.4}$  are super-imposed in both the local and non-local cases.

To this regard, as we have already seen, the probability current field  $\mathbf{j}$  remains bounded in the presence of a vortex line, and thus appears to be a good candidate in order to quantify whether high values of density, as they are observed in particular in the non-local case (Fig. 4 (b)), are associated to high values of the current  $\mathbf{j}$ . We display in Figs. 4 (c) and (d) the histograms of the values taken by the norm of the current vector  $\mathbf{j}$  in the computational domain. We see in both local and non-local cases, that the PDF of  $|\mathbf{j}|$  exhibits similar tendencies, such as (i) a linear trend for  $|\mathbf{j}| \ll 1$ , and (ii) a  $|\mathbf{j}|^{-3}$  power-law behavior in a domain of finite extension, reminiscent of the expected power law of the velocity field

PDF, as explained formerly. The power law behavior at small  $|\mathbf{j}|$  should be a consequence of the interaction of vortices and their images, and the finiteness of the extension of the computational domain (and the respective periodical boundary condition). Even if higher values of the current field  $|\mathbf{j}|$  are indeed observed in the non-local case (Fig. 4 (d)), they do not exceed values already observed in the initial condition at  $t = 0$ . We are led to the conclusion that the existence of a roton minimum has little influence on the overall shape of the PDFs of  $|\mathbf{j}|$ .

Developing on these ideas, we perform a similar estimation of the histogram of the pseudo-vorticity  $\mathbf{w}$  (Eq. 8), in order to quantify possible creation of *small scales*, as it happens in the presence of a direct cascade mechanism, which is at the heart of the phenomenology of Kolmogorov regarding three-dimensional classical (i.e. governed by the viscous Navier-Stokes equation) turbulence [12]. Indeed, recall that in classical turbulence, velocity PDF is close to a Gaussian function, whereas PDF of gradients, and in particular vorticity, is found highly non Gaussian. For a vortex line along the  $z$ -axis, we know well that pseudo-vorticity is expected to be a bounded vector, taking significant values only for  $r$  of the order and smaller than  $a$ . Using thus the schematic distribution provided in Eq. 9, it is easy to get, from Eq. 11, that we expect  $\mathcal{P}_{|\mathbf{w}|}(|\mathbf{w}|) \propto |\mathbf{w}|^{-1-\alpha^{-1}}$ . This assumption on the radial profile of pseudo-vorticity allows to reproduce the present observed histograms of the values taken by  $|\mathbf{w}|$  in our simulation domain at any time of the reconnection process, as it is displayed in Figs. 4 (e) and (f). According to this model, using  $\alpha \approx 2.5$  for both the local and non-local cases, we find that indeed we expect a power law decrease of the PDF with an exponent  $\approx 1.4$ , as it is presently observed. This being said, even if there are some differences implied by the existence of the roton minimum, we are led to the conclusion that there is no creation of small scales, i.e. no creation of high value of pseudo-vorticity, even in the presence of a model taking into account a realistic picture of the core of vortices. We will come back to this point in the conclusion.

## VI. TRACKING VORTICES

Let us now explore some other aspects of the vortex reconnection process, as those related to the evolution of vortices as individual objects. Such kind of studies rely on the tracking of vortex cores, i.e. regions of space where density vanishes. This can be done using algorithms that seek zeros in planes, in order to extract lines in three-dimensional space with the help of the pseudo-vorticity, as it is proposed in recent literature [42]. In this section, we revisit what has been done in this context for the local GP equation [43, 44], and in experiments [45], and compare with what is obtained in the non-local case. In few words, this numerical tracking algorithm provides the position vector  $\mathbf{X}(s, t)$  at each time  $t$  and parameterized by the length  $s$  measured along the filament made of

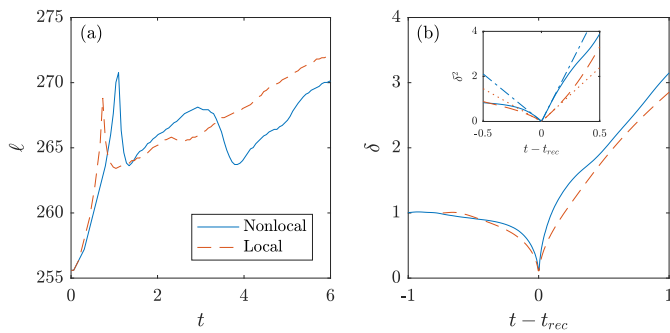


FIG. 5. (a): Total length of the set of two vortices (see text), in the local (red) and non-local (blue) models. (b): Distance  $\delta$  between the two vortices, in the local (red) and non-local (blue). Inset: squared distance  $\delta^2$  between the two vortices for both models and their respective linear fit (dashed and dotted-dashed lines). The slopes of the linear fits before and after reconnection are given in the text.

the points that do not hold a density. From there, we can define the Frenet-Serret frame of reference, given by the orthonormal set of vectors  $(\mathbf{T}, \mathbf{N}, \mathbf{B})$ , where  $\mathbf{T} = \partial \mathbf{X} / \partial s$  is the tangential vector,  $\mathbf{N} = (\partial \mathbf{T} / \partial s) / |\partial \mathbf{T} / \partial s|$  the normal vector, and  $\mathbf{B} = \mathbf{T} \wedge \mathbf{N}$  the binormal vector. In an equivalent way, we could have introduced in the definition of this frame the curvature  $\kappa(s, t) = |\partial \mathbf{T} / \partial s|$  and torsion  $\tau(s, t) = -\mathbf{N} \cdot \partial \mathbf{B} / \partial s$ , where  $\cdot$  stands for the scalar product (see Ref. [46] for instance).

In subsequent developments, we will analyze the vortex reconnection phenomenon in the Frenet-Serret frame, and compare with the well-known schematic model given by the local induction approximation (LIA). LIA has been studied for a long time in various aspects of fluid dynamics, and can be derived in a systematic way from the Navier-Stokes equation [47] assuming that the vortex core size is small compared to some characteristic curvature. From a dynamical point of view, this approximation implies that the time variation of the position  $\mathbf{X}(s, t)$  for some time independent parameterization  $s$  has only a contribution along the binormal vector, namely  $\dot{\mathbf{X}} = \partial \mathbf{X}(s, t) / \partial t = G \kappa(s, t) \mathbf{B}(s, t)$ , where  $G$  is a constant that diverges in a logarithmic fashion with the vortex core size, and  $\kappa(s, t)$  the local curvature. The LIA predicts in a consistent way that indeed the length of such a vortex is conserved, and that solitary waves (i.e. solitons) can propagate at a constant speed [46]. Let us then compare these predictions against the dynamics of the reconnection process that we observe in our numerical estimations of the local and non-local GP equations.

We display in Fig. 5 (a) the length of the system made up of the two vortices, as they are displayed in Fig. 3. We furthermore include in the estimation of their lengths their copies in the whole simulation domain. Remark that the aforementioned tracking algorithm provides in a straightforward manner their lengths. We see that in both local and non-local cases, before reconnection, vor-

tices undergo stretching, that make their length increases of a small amount (of order 5%) before decreasing. It follows then, after reconnection, a monotonic increase for the local case, and a more complex evolution for the non-local case. As we claimed, LIA predicts that the length is a constant a motion. Indeed, defining the length of a vortex as  $\ell(t) = \int |\partial \mathbf{X}(s, t) / \partial s| ds$ , we get from a general point of view (for any parameterization  $s$ ) that  $d\ell(t) / dt = \int \partial \dot{\mathbf{X}} / \partial s \cdot \mathbf{T} ds$ , showing that a non vanishing component of the induced velocity  $\dot{\mathbf{X}}$  along the normal vector  $\mathbf{N}$  may contribute to a variation of the length  $\ell$ , which is not the case in the LIA (only a component along the binormal is considered). Such arguments on vortex length have been rigorously studied in [48]. As we can see in Fig. 5 (a), the length of vortices depends on time, a feature that is not allowed in the LIA, although only 5% of the length undergo changes.

To carry on the description of the dynamical features of reconnection, we compute the distance separating vortices before and after reconnection, which is defined at each time as the minimum distance between two points on the two vortex lines. Such a numerical study has been performed in a systematic fashion for the local GP equation for various initial conditions [44], and it was found that this distance behaves as  $|t - t_{rec}|^{1/2}$ , both before and after reconnection. The square-root behavior can be understood using a linear approach, justified close to the vortex core (where density vanishes) [49], or from a dimensional point of view (see for instance Ref. [50]). The proportionality constant to this square-root law was found to depend on initial conditions, whether, as an example, vortices are taken perpendicular or anti-parallel. We represent in Fig. 5 (b) the time evolution of this distance  $\delta$  before and after reconnection in our present numerical simulations (see also a representation of  $\delta^2$  in the inset). For both the local and non-local cases, we reproduce the square-root behavior close to the reconnection time  $t_{rec}$ , but with slightly different proportionality constants. This numerical estimation shows that the roton minimum has here some influence, in particular we see that approaching and separation distances have different time evolution. The numerical values we observe for the proportionality constant in the local case (resp. nonlocal) are significantly different from the values observed in ref. [44]. We find 0.23 (before reconnection) and 0.38 (after reconnection) as far the local case is considered. Concerning the nonlocal case, we find 0.33 (before  $t_{rec}$ ) and 0.81 (after  $t_{rec}$ ). We recall that it was found in Ref. [44], for the local case, the corresponding values 0.55 and 0.63. Remark that, in this study, the two orthogonal vortices are initially separated by an atomic distance  $a$ , whereas this initial distance was chosen to be six healing lengths (which is of the same order of  $a$ ), thus a factor of order 6 for the chosen initial separations. This could explain once again the strong dependence of this multiplicative constant on initial conditions, and the differences in between the present numerical study and the one proposed in Ref. [44].



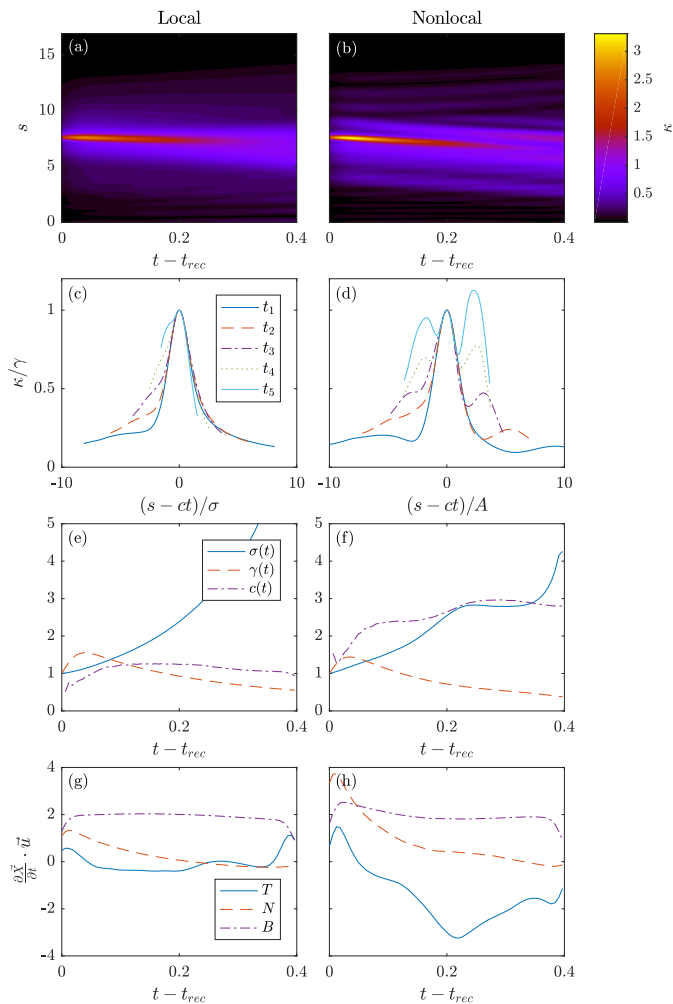


FIG. 6. (a,b): space-time maps of the local curvature of a vortex line  $\kappa(s,t)$ , in the local and non-local models respectively. The position of the principal curvature maximum is tracked along time. (c,d): curvature profiles of the tracked maximum at five regularly spaced times during its propagation. Each profile is scaled back onto the initial condition at reconnection. Snapshot times are  $t_1 = t_{\text{rec}}$ ,  $t_2 = t_{\text{rec}} + 0.09$ ,  $t_3 = t_{\text{rec}} + 0.17$ ,  $t_4 = t_{\text{rec}} + 0.26$  and  $t_5 = t_{\text{rec}} + 0.34$ . We observe the growth of secondary curvature maxima in the non-local case (d). (e,f): time evolution of the  $A$ ,  $B$ ,  $c$  scaling parameters, which represent the width, height and velocity of the curvature soliton respectively (see Eq. 12). (g,h): projections of the velocity vector of the local maximum of curvature onto the Frenet-Serret orthonormal basis  $\mathbf{T}$ ,  $\mathbf{N}$ ,  $\mathbf{B}$ .

## VII. CHARACTERIZATION OF A PROPAGATING SOLITON

Visualization of the overall reconnection phenomenon, as it is displayed in Fig. 3 suggests the creation of a localized phenomenon along the vortices. To exhibit in a quantitative fashion this phenomenon, we display in Figs. 6 (a) and (b) maps of local curvature, i.e. the numerical estimation, thanks to the vortex tracking algorithm,

of spatiotemporal maps of  $\kappa(s,t)$  as a function of the arc-length  $s$  and time  $t$ . We indeed observe in both the local and non-local cases the propagation of a soliton of curvature, as it is predicted by the LIA, and conveniently formalized in Ref. [46]. Remark first that indeed, the soliton, and in particular the local maximum of curvature (i.e. bright colors), seems to propagate in an approximate fashion at a constant speed i.e.  $c = ds/dt$ , for both the local and non-local cases, although in the non-local case, secondary maxima appear during the propagation.

To quantify more accurately the soliton velocity and its shape in the vicinity of the curvature maximum, we display in Figs. 6 (c) and (d) a tentative rescaling of the observed curvature at various instants as

$$\gamma(t)\kappa\left(\frac{s-c(t)t}{\sigma(t)}, t\right), \quad (12)$$

where  $c(t)$  is the velocity of the maximum of curvature,  $\sigma(t)$  a typical length quantifying the increase in the width of the soliton, and  $\gamma(t)$  that allows to include a possible time-variation in the amplitude. We indeed observe that this rescaling procedure makes the curvature profile similar to what is observed initially. Under the LIA, it is shown in Ref. [46] that such a soliton is expected to behave as a solitary wave that propagates at a constant velocity (given by the initial torsion) and does not change neither its shape, nor its amplitude, i.e.  $\sigma(t) = \gamma(t) = 1$ , the actual time independent shape being given by a hyperbolic secant function. In our present numerical simulation, we see that in a good approximation, except at early times ( $t < 0.1$ ), the velocity of the soliton,  $c(t)$ , is nearly constant, of order  $c \approx 1$  in the local case, and of order  $c \approx 2$  in the non-local one, as it is suggested in the LIA approach. In comparison, the celerity of sound in the superfluid is  $c_s = 16$  in these units: the observed soliton propagates in a much slower way than acoustic waves. On the contrary, the soliton undergoes both dispersion, i.e. its width increases as tracked by the increase of the rescaling coefficient  $\sigma(t)$ , and its amplitude decreases, with accordingly a decrease in the coefficient  $\gamma(t)$ . A more precise analysis shows indeed that  $\gamma(t) \sim 1/t$ , a decrease that is not predicted by LIA.

In order to interpret these observed behaviors, we display in Figs. 6 (g) and (h) the projections of the vortex velocity vector  $\partial\mathbf{X}/\partial t$  in the Frenet-Serret frame of reference ( $\mathbf{T}$ ,  $\mathbf{N}$ ,  $\mathbf{B}$ ) as a function of time. We indeed observe that the projection along the binormal vector is constant during the soliton propagation, as it is assumed in the LIA. Let us precise here that if indeed the projection of  $\partial\mathbf{X}/\partial t$  on  $\mathbf{B}$  appears to be time independent, we can infer from the tracking of the soliton itself (Figs. 6 (c) and (d)) that the actual value of this projection cannot be given by only the curvature since curvature itself is found dependent on time. Interestingly, at early stage following reconnection, once again for  $t < 0.1$ , we see a non vanishing contribution along the normal vector  $\mathbf{N}$ . As shown in Ref. [48], this part of the dynamics is involved in a self-stretching phenomenon that would modify the length of

vortices, as it is indeed observed in Fig. 5 (a). As time evolves, this projection gets smaller and smaller. Finally, we see that in the local case, the projection along the tangential vector  $\mathbf{T}$  is always negligible in front the projections along the other direction of the frame, whereas it cannot be neglected in the non-local case. This might be explained while studying the interaction with additional local maxima that appear during the soliton propagation for the non-local case.

### VIII. CONCLUSION AND FINAL REMARKS

We have studied numerically the reconnection phenomenon of two initially orthogonal quantum vortices in a model of superfluids which includes the roton minimum in the dispersion relation. As it was proposed in the literature [30, 31, 40], such a model describes the dynamics of a scalar wave function that is governed by the Gross-Pitaevskii equation [26] where is considered a non-local two-body interaction of characteristic extension the  $^4\text{He}$  atomic size  $a$ . We start by calibrating the model to be as close as possible to the experimental dispersion relation of  $^4\text{He}$  provided in Ref. [29], with the additional will to prevent from the generation of precursors of crystallization, as it was evidenced in Ref. [40], although of great importance in the context of supersolidity [34, 35]. Once obtained such a model, we estimate its stationary vortex solution, that is a time independent solution with an axis symmetry, around which the phase turns of an angle equals to  $2\pi$ . Then, in a similar way as in Ref. [27], we prepare a initial condition made up of two of these vortices in a orthogonal situation. We indeed observe a reconnection, and track and study the time evolution of the density, current and pseudo-vorticity fields, we provide both a statistical analysis of the fields and a local estimation of the geometry of these vortices, including a precise characterization of a observed soliton that shares some features predicted by the LIA.

This numerical investigation shows that taking into account a more realistic structure of vortices, as depicted in Ref. [32], has little influence on the global picture of reconnection given by the local version of the GP equation and presented in Ref. [27], although the creation and propagation of a soliton of curvature along the vortex cores appear more complex when the non-local interaction is plugged in the dynamics.

The statistical analysis of the kinematic quantities involved in the dynamics, in particular current  $\mathbf{j}$  and pseudo-vorticity  $\mathbf{w} = \nabla \wedge \mathbf{j}$ , shows that there is no creation of scales smaller that the injected atomic length size  $a$ . This makes a big difference with what is obtained with the incompressible Euler or Navier-Stokes equations, where a cascading phenomenon transfers energy towards the small scales, as recently put in evidence while considering two colliding vortex rings in a experimental (classical) flow [51]. We can thus infer that the hydrodynamics implied by the local and non-local versions of the GP equations, because of its implied high level of compressibility in the vicinity of the vortices, and the unclear action of the additional quantum pressure term, is indeed very different from the one of incompressible viscous Newtonian fluids.

It would of tremendous importance to develop an interaction term in the GP evolution of the wave function able to include a more realistic prediction of the dispersion relation, without exhibiting crystallization phenomena that are not expected in the superfluid phase of  $^4\text{He}$ . We keep this perspective for future investigations.

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