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Abstract. We propose linear dynamics that can generate a given sea wavenumber spectrum via a linear partial differential equation stirred by an additive random forcing term that is δ-correlated in time. In particular, the correlation structure of the solution to this linear dynamics converges towards the target spectrum as time passes. The main linear mechanism is a transport in Fourier space, which models the transfer of energy from large scales towards small scales. The proposed linear dynamics generalize previous works for power-law spectra in the context of hydrodynamic turbulence to more general spectra, possibly non-radial, including sea wavenumber spectra such as the JONSWAP spectrum. Finally, we present simulations of the correlation structure of the solution to these dynamics in 2D, whose spectrum converges towards the JONSWAP spectrum. These simulations are based on a finite volume method in the Fourier domain and a splitting method in time, following a recently proposed numerical method by the same authors which improves on pseudospectral simulations.

1. Introduction

1.1. **Background.** Turbulence is often understood as the intricate motion of vortices in fluid dynamics when the Reynolds number is large. From a phenomenological point of view, one of the main objects of study is the transfer of energy from such vortices to smaller and smaller vortices. This is known as energy cascade, a widely studied phenomenon since the pioneering work of Richardson, Kolmogorov and others [19, 10].

In a more general setting, the word turbulence also refers to the chaotic behavior of systems having a large number of degrees of freedom. In the case of linear water wave systems, the surface elevation can be decomposed into independent oscillating waves (Fourier modes) whose evolution can be exactly computed. Gravity water waves equations are however nonlinear and thus Fourier modes do not evolve independently, but they interact with each other, generating new Fourier modes as time passes. Some of these interactions allow for the transfer of energy to higher and higher wavenumbers, which is reminiscent of Kolmogorov's work on energy cascades.

Due to the large number of interactions between these Fourier modes, a deterministic description of the evolution of the energy is not suitable and a stochastic approach is needed. Thus the natural object in wave turbulence is not the evolution of each individual mode but of its statistical average. Averaging allows one to carry out a kinetic theory of colliding waves in the same fashion as Boltzmann's kinetic theory for particles.

In the 1960s, Hasselmann heuristically derived a wave-kinetic equation for gravity water waves, which describes the effective evolution of the average energy of each Fourier mode of the system [12, 13]. At the same time, Zakharov made a connection between hydrodynamic turbulence and the interaction of waves in the general context of nonlinear dispersive Hamiltonian PDEs. Thus the kinetic theory for colliding waves became known as wave turbulence. The first rigorous derivation of a wave-kinetic equation was obtained by Deng-Hani [7, 8] in the context of the cubic nonlinear Schrödinger equation (see also [5, 6, 11, 18]) but analogous results for the gravity water waves system are not yet available due, in no small part, to its quasilinear nature. Even if rigorous derivation is not avaible, such wave kinetic equations are used in practical for weather forecasts by météo france

(Wavewatch III, see [20]). Zakharov found stationary solutions to such wave kinetic equations which capture the mean energy transfer from small wavenumbers towards large wavenumbers once the wave system reaches a statistically stationary state. Such stationary solutions, also known as spectra, represent the correlation structure of the energy of the wave system.

Statistically homogeneous fields are a fundamental object in the phenomenology of homogeneous and isotropic turbulent flows [3, 10, 17]. In this idealized situation, the underlying probability law of the turbulent velocity field is invariant by spatial translations and rotations, and has been repeatedly observed in laboratory flows and in simulations of the forced Navier-Stokes equations. Given a spatial field u which is statistically homogeneous, we call its wave number spectrum the function $S(k)$ such that

$$\mathbb{E}[\hat{u}(k)\overline{\hat{u}(k')}] = S(k)\,\delta_{k-k'}\ ,$$

where $\hat{u}$ is the spatial Fourier transform of u, k is the wave number and $\delta_{k-k'}$ stands for the Dirac delta distribution. In the context of the velocity field of a fluid with a large Reynolds number, it is experimentally observed that the fluid reaches a statistically stationary, homogeneous and isotropic state in the absence of boundary. In such cases, the wave number spectrum of the Eulerian velocity only depends on $|k|$. More precisely, according to Kolmogorov's 41 theory [15], the wavenumber spectrum is a curve which may be roughly decomposed into three regions:

- Injection range: for small $|k|$, the shape of the curve $S(k)$ is determined by the forcing.
- Inertial range: for intermediate values of $|k|$, energy is transferred from large wavenumbers towards small wavenumbers. In this range, the spectrum is observed (and predicted by Kolmogorov's theory) to be a power law of the form $|k|^{-d-2H}$ with $H \approx 1/3$.
- Dissipative range: for large $|k|$, viscous dissipation damps the energy coming from small wavenumbers, effectively smoothing the spatial velocity profile.

Sea wavenumber spectra are often written as follows

$$S(k) = S_{\mathrm{r}}(\omega(k))S_\theta(k), \tag{1.1}$$

where $\omega(k)$ is the dispersion relation of the system [16]. For gravity water waves in the ocean with infinite depth, the dispersion relation is $\omega(k) = \sqrt{g|k|}$ where g is the gravity, whereas one has $\omega(k) = g|k|^2$ in the shallow water regime. Different dispersion relations are possible if one considers surface tension or other phenomena.

The angular part $S_\theta^f(k)$ of the spectrum (1.1) typically describes the energetic spreading of the waves as a function of the wind direction [2, 9]. The most widely used radial spectrum $S_{\mathrm{r}}(\omega(\cdot))$ for fully developped sea is the JONSWAP (Joint North Sea Wave Observation Project) spectrum:

$$S_{\mathrm{r}}(\omega(k)) = \frac{\alpha g^2}{\omega(k)^5}\exp\left[-\frac{5}{4}\frac{\omega_p^4}{\omega(k)^4}\right]\gamma^{\exp\left[-\frac{(\omega(k)-\omega_p)^2}{2\sigma^2\omega_p^2}\right]} \tag{1.2}$$

where $\gamma = 3.3$, and the parameters α, ω_p, σ depend on the speed of the wind and its fetch [14]. The Pierson-Moskowitz spectrum corresponds to $\gamma = 1$. The choice $\gamma = 1$ and $\omega_p = 0$ leads to the Phillips spectrum[1], which describes white capping. In intermediate range of $|k|$, the JONSWAP spectrum $S_{\mathrm{r}}(\omega(k)) \sim \omega(k)^{-4}$ which corresponds to the Zakharov stationary solutions to the wave kinetic equation associated to the water waves equations, also known as the Hasselmann spectrum.

A great number of wave spectra are based on observations and do not (yet) admit a rigorous mathematical justification. In particular, in the context of wave turbulence, there is no rigorous

[1]Kuznetsov argues that the Phillips spectrum in deep water should be taken with $\omega \sim |k|$ instead of $\omega \sim \sqrt{|k|}$, as the most common slope breaks occur on 1D lines/ridges, not on 0D point/peaks, and that wave crests are propagating with preserved shape.

derivation of such spectra for the gravity water waves system. Similarly, in the context of hydrodynamic turbulence, no mathematical proof of the regimes described in Kolmogorov's theory exists starting from the Navier-Stokes equation in 3D. For this reason, the construction of toy models which feature such spectra is of fundamental importance. A large number of such models are static, in the sense that the model is already in a statistically stationary state featuring the desired spectrum. Other models are dynamic but only in the form of discrete dynamical systems.

The main result of this paper is the construction of a PDE driven by a random additive force whose solution converges to a statistically stationary state featuring the desired spectrum.

1.2. **Main results.** Before we present our model, we introduce the main hypotheses on the target spectrum and the forcing.

The forcing term $(t,x) \mapsto \xi^\varphi(t,x)$ is a statistically homogeneous and stationary field whose correlations satisfy

$$\mathbb{E}[\xi^\varphi(t,x)\xi^\varphi(s,x')] = C_\varphi(x-x')\,\delta_{t-s} \qquad \forall t,s \geq 0,\ x,x' \in \mathbb{R}^d,$$

where δ is the delta Dirac distribution and

$$C_\varphi(x) = \int_{\mathbb{R}^d_x} \varphi(x+y)\varphi(y)\,\mathrm{d}y, \quad \forall\, x \in \mathbb{R}^d.$$

Note that the correlations of the forcing term are distributions. In Section 2.1, we construct such a forcing term as the spatial convolution of a space-time white noise with the function φ. We make the following assumptions on the latter.

Assumption 1.1. *The function φ is a real-valued, spectrally supported function. More precisely, we assume that its Fourier transform $\widehat{\varphi}$ is continuous and compactly supported away from the origin, i.e. there exist $\kappa_\varphi > \kappa > 0$ such that*

$$\widehat{\varphi}(k) = 0 \text{ if } |k| < \kappa \text{ or } |k| > \kappa_\varphi.$$

Given a fixed wavenumber spectrum $k \mapsto S(k)$, we make the following assumptions:

Assumption 1.2. *$S : \mathbb{R}^d \to \mathbb{R}$ is a continuous function which is strictly positive in the region $\mathbb{R}^d \setminus \{0\}$. We further assume that $k \mapsto S(k)$ is integrable in the region $|k| > \kappa > 0$, with κ defined in Assumption 1.1. Moreover, for every $a \geq 0$, the following property holds:*

$$\sup_{\substack{k\in\mathbb{R}^d,\\ |k|\geq\kappa}} \frac{S\left(\frac{|k|+a}{|k|}k\right)}{S(k)} < \infty. \tag{1.3}$$

We are ready to introduce our PDE in wavenumber space:

$$\begin{cases} \partial_t \widehat{u}(t,k) + c\,\partial_{|k|}\,\widehat{u}(k) + cV(k)\,\widehat{u}(t,k) = \widehat{\xi^\varphi}(t,k) & t>0, k\in\mathbb{R}^d, |k|>\kappa>0, \\ \widehat{u}(t,k) = 0 & t>0, k\in\mathbb{R}^d, |k|\leq\kappa, \\ \widehat{u}(0,k) = 0. \end{cases} \tag{1.4}$$

where $\partial_{|k|} = \frac{k}{|k|}\cdot\nabla_k$ stands for the transport with respect to the radial variable and

$$V(k) = \partial_{|k|}\log(\sigma(k)) \quad \text{where} \quad \sigma(k) = |k|^{\frac{d-1}{2}} S(k)^{-\frac{1}{2}}. \tag{1.5}$$

Note that (1.4) must be interpreted in a weak sense since the source $\widehat{\xi^\varphi}(t,k)$ is delta-correlated in t and k. An important part of this work is thus to give a rigorous meaning to (1.4) and to define a notion of solution.

Our main goal is to show that the solution to (1.4) exhibits the desired spectrum $S(k)$ as $t \to \infty$. The main result of this paper can be formally formulated as follows:

Formal Theorem (Description of the spectrum)**.** *Under Assumption 1.1 and Assumption 1.2, there exists a solution to* (1.4) *whose inverse Fourier transform is a well-defined Gaussian field. Moreover, the solution has the following asymptotic correlations:*

$$\lim_{t\to\infty} \mathbb{E}[\widehat{u}(t,k)\overline{\widehat{u}(t,k')}] = S(k)\,\vartheta(k)\,\delta_{k-k'} \tag{1.6}$$

where

$$\vartheta(k) = \mathbb{1}_{|k|>\kappa} \int_\kappa^{|k|} \left|\widehat{\varphi}\left(s\frac{k}{|k|}\right)\right|^2 \frac{1}{S\left(s\frac{k}{|k|}\right)} \frac{\mathrm{d}s}{c}. \tag{1.7}$$

A mathematically rigorous statement of this theorem may be found in Proposition 2.2 (in Fourier space), Proposition 2.4 (in physical space) and Corollary 2.9 (asymptotic correlations). An example of energy cascades in the case $S(k) = |k|^{-(d+2H)}$ may be found in Proposition 2.10.

In Section 4, we present numerical simulations to illustrate the results in this theorem in the case where $S(k)$ is the JONSWAP spectrum (1.2).

Remark 1.3. *If the spectrum S is radial away from the injection range, namely $S(k) = S_{\mathrm{r}}(|k|)$ for all $|k| > \kappa_f$, then ϑ in* (1.7) *only depends on the angular variable away from the injection range, i.e.*

$$\vartheta(k) = S_\theta\left(\frac{k}{|k|}\right) := \int_\kappa^{\kappa_\varphi} \left|\widehat{\varphi}\left(s\frac{k}{|k|}\right)\right|^2 \frac{1}{S\left(s\frac{k}{|k|}\right)} \frac{\mathrm{d}s}{c}, \qquad \forall\ |k| > \kappa_\varphi.$$

Remark 1.4. *If the spectrum S and $\widehat{\varphi}$ are radial, then $\vartheta(k) = \mathfrak{C}$ is constant for all $|k| > \kappa_\varphi$, where*

$$\mathfrak{C} = \int_\kappa^{\kappa_\varphi} \frac{|\widehat{\varphi}(s\gamma)|^2}{c\,S\,(s\gamma)}\ \mathrm{d}s,$$

for an arbitrary unit vector $\gamma \in \mathbb{S}^{d-1}$. In particular, this means that away from the injection range the solution $\widehat{u}$ to (1.4) *has the following spectrum:*

$$\lim_{t\to\infty} \mathbb{E}[\widehat{u}(t,k)\overline{\widehat{u}(t,k')}] = \mathfrak{C}\,S(k)\,\delta_{k-k'} \qquad \forall\ |k| > \kappa_\varphi.$$

Remark 1.5. *For a continuous function $k \in \mathbb{R}^d \mapsto S(k) \in \mathbb{R}$ which is strictly positive in $\mathbb{R}^d \setminus \{0\}$, it suffices to check* (1.3) *for a sufficiently large $|k| > K_a$. If S is a radial function, a sufficient condition for* (1.3) *to hold is for S to be non-increasing. Combining these observations,* (1.3) *is easily shown to hold in the case of the JONSWAP spectrum* (1.2)*.*

1.3. **Outline.** In Section 1.4 we introduce the notation used throughout this article. In Section 2 we present a rigorous analysis of the model (1.4) in Fourier and physical space, including detailed definitions of the forcing and notion of solution. In particular, Section 2.3 contains the asymptotic analysis of the correlations, including the proof of (1.6). In Section 3 we present the numerical method used to conduct numerical simulations of our model (1.4). Finally, in Section 4 we present and discuss said simulations in the cas of the JONSWAP spectrum in dimension $d = 2$.

1.4. **Notation.** Given two functions $\phi, \phi' \colon \mathbb{R}^d \to \mathbb{C}$, define the convolution product $\phi * \phi'$ and the correlation product $\phi \star \phi'$ by

$$\phi * \phi'(x) = \int_{\mathbb{R}^d} \phi(x-y)\phi'(y)\,\mathrm{d}y, \qquad \phi \star \phi'(x) = \int_{\mathbb{R}^d} \phi(x+y)\phi'(y)\,\mathrm{d}y.$$

One has

$$\int_{\mathbb{R}^d} (\varphi * \phi)\phi' = \int_{\mathbb{R}^d} \phi(\varphi \star \phi') \tag{1.8}$$

for any function $\varphi, \phi, \phi' \colon \mathbb{R}^d \to \mathbb{C}$ such that the above integrals are well-defined.

The indicator function of a set B is denoted by $\mathbb{1}_B$. We let $\mathbb{R}^+ = [0, \infty)$.

Given $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let $L^2(\mathbb{R}^d_x; \mathbb{K})$, resp. $L^2(\mathbb{R}^+_t \times \mathbb{R}^d_x; \mathbb{K})$, denote the spaces of square-integrable functions $\phi \colon x \in \mathbb{R}^d \mapsto \phi(x) \in \mathbb{K}$, resp. $\phi \colon (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \mapsto \phi(t,x) \in \mathbb{K}$. The inner products in $L^2(\mathbb{R}^d_x; \mathbb{C})$ and $L^2(\mathbb{R}^+_t \times \mathbb{R}^d_x; \mathbb{C})$ are denoted by $(\cdot,\cdot)_{L^2_x}$ and $(\cdot,\cdot)_{L^2_{t,x}}$ and defined by

$$(\phi, \phi')_{L^2_x} = \int_{\mathbb{R}^d} \phi(x)\overline{\phi'(x)}\,\mathrm{d}x, \quad \forall\ \phi, \phi' \in L^2(\mathbb{R}^d_x; \mathbb{C}),$$
$$(\Phi, \Phi')_{L^2_{t,x}} = \int_0^{+\infty} \int_{\mathbb{R}^d} \phi(t,x)\overline{\phi'(t,x)}\,\mathrm{d}t\,\mathrm{d}x, \quad \forall\ \Phi, \Phi' \in L^2(\mathbb{R}^+_t \times \mathbb{R}^d_x; \mathbb{C}),$$

and the associated norms are denoted by $\|\cdot\|_{L^2_x}$ and $\|\cdot\|_{L^2_{t,x}}$ respectively.

The Fourier transform of an integrable function $\phi \colon \mathbb{R}^d_x \to \mathbb{C}$ is defined by

$$\widehat{\phi}(k) = \mathcal{F}\phi(k) := \int_{\mathbb{R}^d_x} e^{-2\pi i k\cdot x}\phi(x)\,\mathrm{d}x, \quad \forall\ k \in \mathbb{R}^d.$$

Whenever defined, the inverse Fourier transform of an integrable function $\widehat{\phi} \colon \mathbb{R}^d_k \to \mathbb{C}$ is

$$\phi(x) = \widehat{\phi}^{\vee}(x) = \mathcal{F}^{-1}\widehat{\phi}(x) := \int_{\mathbb{R}^d_k} e^{2\pi i x\cdot k}\,\widehat{\phi}(k)\,dk, \quad \forall\ x \in \mathbb{R}^d.$$

It is well know that the Fourier transform is an isometry from $L^2(\mathbb{R}^d_x)$ to $L^2(\mathbb{R}^d_k)$:

$$(\widehat{\phi}, \widehat{\phi'})_{L^2_k} = (\phi, \phi')_{L^2_x}, \quad \forall\ \phi, \phi' \in L^2(\mathbb{R}^d_x; \mathbb{C}).$$

Let $\big(\Omega, \mathcal{F}, \mathbb{P}\big)$ denote the probability space. Given $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let $L^2(\Omega; \mathbb{K})$ be the space of square-integrable random variables with values in $\mathbb{K}$.

The equation studied in this article is driven by Gaussian real-valued space-time white noise $\xi(t,x)$ for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$. Formally, this Gaussian noise is characterized by its mean $\mathbb{E}[\xi(t,x)] = 0$ and its covariance

$$\mathbb{E}[\xi(t,x)\xi(t',x')] = \delta(t-t')\delta(k-k'),$$

where δ stands for the Dirac pulse distribution. The formulation above is formal and shows that white noise needs to be understood in a distributional sense. A more rigorous interpretation consists in considering a Gaussian process

$$\begin{array}{rccc} \Xi\colon & L^2(\mathbb{R}^+_t \times \mathbb{R}^d_x; \mathbb{R}) & \to & L^2(\Omega; \mathbb{R}), \\ & \Phi & \mapsto & \Xi(\Phi) = \langle \xi, \Phi \rangle, \end{array}$$

that satisfies

- for any $N \in \mathbb{N}$ and any $\Phi_1, \dots, \Phi_N \in L^2(\mathbb{R}^+_t \times \mathbb{R}^d_x; \mathbb{R})$, $\big(\langle \xi, \Phi_1\rangle, \dots, \langle \xi, \Phi_N\rangle\big)$ is a $\mathbb{R}^d$-valued Gaussian random variable,

- for any $\Phi \in L^2(\mathbb{R}_t^+ \times \mathbb{R}_x^d; \mathbb{R})$,
$$\mathbb{E}[\langle \xi, \Phi \rangle] = 0,$$
- for any $\Phi, \Phi' \in L^2(\mathbb{R}_t^+ \times \mathbb{R}_x^d; \mathbb{R})$,
$$\mathbb{E}[\langle \xi, \Phi \rangle \langle \xi, \Phi' \rangle] = (\Phi, \Phi')_{L^2_{t,x}}.$$

Such a Gaussian process is called an isonormal Gaussian process.

Setting
$$\langle \xi, \Phi \rangle = \langle \xi, \mathrm{Re}(\Phi) \rangle + i \langle \xi, \mathrm{Im}(\Phi) \rangle$$
for any complex-valued square-integrable function $\Phi \in L^2(\mathbb{R}_t^+ \times \mathbb{R}_x^d; \mathbb{C})$ extends the definition of ξ as an isonormal Gaussian process defined from $L^2(\mathbb{R}^+ \times \mathbb{R}^d; \mathbb{C})$ to $L^2(\Omega; \mathbb{C})$.

The Gaussian random variable $\Xi(\Phi) = \langle \xi, \Phi \rangle$ is sometimes written as a stochastic Wiener integral
$$\langle \xi, \Phi \rangle = \int_0^{+\infty} \int_{\mathbb{R}^d} \Phi(t,x) \, \mathrm{d}\xi(t,x).$$

Note that it is sufficient to consider functions $\Phi = \psi \otimes \phi$, such that $\Phi(t,x) = \psi(t)\phi(x)$, given square-integrable functions $\psi \in L^2(\mathbb{R}_t^+; \mathbb{C})$ and $\phi \in L^2(\mathbb{R}_x^d; \mathbb{C})$. It is then possible to define the Fourier transform $\widehat{\xi}(t,k)$ of the space-time white noise $\xi(t,x)$ as follows: for any $\psi \in L^2(\mathbb{R}_t^+; \mathbb{C})$ and $\phi \in L^2(\mathbb{R}_k^d; \mathbb{C})$, set
$$\langle \widehat{\xi}, \psi \otimes \phi \rangle = \langle \xi, \psi \otimes \widehat{\phi} \rangle.$$
The notation $\widehat{\xi}(t,k)$ is formal, since this is a white noise, with mean $\mathbb{E}[\widehat{\xi}(t,k)] = 0$ and the covariance
$$\mathbb{E}[\widehat{\xi}(t,k)\overline{\widehat{\xi}(t',k')}] = \delta(t-t')\delta(k-k').$$
The above expression is a formal version of the property
$$\begin{aligned}\mathbb{E}[\langle \widehat{\xi}, \psi \otimes \phi \rangle \overline{\langle \widehat{\xi}, \psi' \otimes \phi' \rangle}] &= \mathbb{E}[\langle \xi, \psi \otimes \widehat{\phi} \rangle \langle \xi, \psi' \otimes \widehat{\phi'} \rangle] \\ &= (\psi \otimes \widehat{\phi}, \psi' \otimes \widehat{\phi'})_{L^2_{t,x}} \\ &= (\psi \otimes \phi, \psi' \otimes \phi')_{L^2_{t,k}}\end{aligned}$$
which follows from the definition of the Fourier transform of the white noise and of the isometry property of the Fourier transform.

Being the Fourier transform of a real-valued white noise $\xi(t,x)$, a property $\overline{\widehat{\xi}(t,k)} = \widehat{\xi}(t,-k)$ is satisfied, in a weak sense, therefore it is not necessary to deal with $\mathbb{E}[\widehat{\xi}(t,k)\widehat{\xi}(t',k')]$ in the description of $\widehat{\xi}(t,k)$.

Define the subspace $\widehat{\mathcal{H}}$ of $L^2(\mathbb{R}_k^d)$ by
$$\widehat{\mathcal{H}} = \{\widehat{f} \in L^2(\mathbb{R}_k^d) \mid \widehat{f}(k) = 0 \text{ if } |k| < \kappa\}. \tag{1.9}$$
Let also $\mathcal{H}$ be the subspace of $L^2(\mathbb{R}_x^d)$ defined by
$$\mathcal{H} = \{f \in L^2(\mathbb{R}_x^d) \mid \widehat{f} \in \widehat{\mathcal{H}}\}. \tag{1.10}$$

Let us mention a useful change of variables formula which is used several times below: given a function $F : (k_1, k_2) \in \mathbb{R}_k^d \times \mathbb{R}_k^d \mapsto F(k_1, k_2) \in [0, \infty]$, one has
$$\int_{\mathbb{R}^d} F\left(k, \frac{|k| - ct}{|k|} k\right) \mathbb{1}_{|k|-ct>\kappa} \, \mathrm{d}k = \int_{\mathbb{R}^d} \left(\frac{|k| + ct}{|k|}\right)^{d-1} F\left(\frac{|k| + ct}{|k|} k, k\right) \mathbb{1}_{|k|>\kappa} \, \mathrm{d}k. \tag{1.11}$$

The identity (1.11) is due to a change of variables in spherical coordinates, letting $r = |k| \in \mathbb{R}^+$ and $\theta = \frac{k}{|k|} \in \mathbb{S}^{d-1}$ for all $k \in \mathbb{R}^d$. If μ denotes the surface measure on the sphere $\mathbb{S}^{d-1}$, using successively the changes of variables $k \mapsto (r,\theta) = (|k|, \frac{k}{|k|})$, $(r,\theta) \mapsto (r',\theta) = (r-ct,\theta)$ and $(r',\theta) \mapsto k' = r'\theta$, one obtains

$$\begin{aligned}\int_{\mathbb{R}^d} F\left(k, \frac{|k|-ct}{|k|}k\right) \mathbb{1}_{|k|-ct>\kappa}\,\mathrm{d}k &= \int_{\mathbb{R}^+\times\mathbb{S}^{d-1}} F(r\theta,(r-ct)\theta)\,\mathbb{1}_{r-ct>\kappa}\,r^{d-1}\,\mathrm{d}r\,\mathrm{d}\mu(\theta)\\ &= \int_{\mathbb{R}^+\times\mathbb{S}^{d-1}} F((r'+ct)\theta,r'\theta)\,\mathbb{1}_{r'>\kappa}\,(r'+ct)^{d-1}\,\mathrm{d}r\,\mathrm{d}\mu(\theta)\\ &= \int_{\mathbb{R}^d}\left(\frac{|k'|+ct}{|k'|}\right)^{d-1} F\left(\frac{|k'|+ct}{|k'|}k',k'\right)\mathbb{1}_{|k'|>\kappa}\,\mathrm{d}k'.\end{aligned}$$

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2. Analysis of the model

2.1. **In the Fourier domain.** The objective of this section is to give a meaning to the model when considered in the Fourier domain. This formulation will be considered for the numerical approximation. Looking for the solution as a Gaussian random field $(t,k) \in \mathbb{R}^+\times\mathbb{R}^d \mapsto \widehat{U}(t,k)$ is not convenient, indeed there is no regularization mechanism in the dynamics, and at any time $t \geq 0$ the mapping $k \mapsto \widehat{U}(t,k)$ would have the same irregularity of white noise: more precisely one would have

$$\mathbb{E}[\widehat{U}(t,k)\overline{\widehat{U}(t',k')}] = \Gamma(t,k)\delta(k-k'),$$

for a function $(t,k) \in \mathbb{R}^+\times\mathbb{R}^d \mapsto \Gamma(t,k)$. As in the description of the white noise $\xi(t,x)$ and of its Fourier transform $\widehat{\xi}(t,k)$ in Section 1.4, it is more convenient to consider, at each time $t \geq 0$, the Gaussian random variable $\widehat{U}(t) : L^2(\mathbb{R}^d_k;\mathbb{C}) \to L^2(\Omega;\mathbb{C})$, i.e. the complex-valued random variables $\langle\widehat{U}(t),\phi\rangle$ for all $\phi \in L^2(\mathbb{R}^d_k;\mathbb{C})$. It is even sufficient to deal with functions ϕ that belong to the subspace $\widehat{\mathcal{H}}$. In the sequel, we show how $\widehat{U}(t)$ is defined and we justify the following identity: for all $t \in \mathbb{R}^+$ and all $\phi,\phi' \in \widehat{\mathcal{H}}$ one has

$$\mathbb{E}[\langle\widehat{U}(t),\phi\rangle\overline{\langle\widehat{U}(t),\phi'\rangle}] = \int_{\mathbb{R}^d}\Gamma(t,k)\phi(k)\overline{\phi'(k)}\,\mathrm{d}k.$$

In order to define the process $\big(\widehat{U}(t)\big)_{t\geq0}$, several ingredients need to be introduced: a Wiener process $\big(\widehat{W}(t)\big)_{t\geq0}$, the linear operator $\widehat{A}$, the semigroup $\big(\widehat{\Pi}_t\big)_{t\geq0}$ and its adjoint $\big(\widehat{\Pi}_t^\star\big)_{t\geq0}$.

For any $t \in \mathbb{R}^+$ and any $\phi \in L^2(\mathbb{R}^d_k)$, set

$$\langle\widehat{W}(t),\phi\rangle = \langle\widehat{\xi},\mathbb{1}_{[0,t]}\otimes\phi\rangle. \tag{2.1}$$

where $\big(\mathbb{1}_{[0,t]}\otimes\phi\big)(s,k) = \mathbb{1}_{[0,t]}(s)\,\phi(k)$. For all $t,t' \in \mathbb{R}^+$ and any $\phi,\phi' \in L^2(\mathbb{R}^d_k)$, it follows that

$$\begin{aligned}\mathbb{E}[\langle\widehat{W}(t),\phi\rangle\overline{\langle\widehat{W}(t'),\phi\rangle}] &= \mathbb{E}[\langle\widehat{\xi},\mathbb{1}_{[0,t]}\otimes\phi\rangle\overline{\langle\widehat{\xi},\mathbb{1}_{[0,t']}\otimes\phi'\rangle}]\\ &= (\mathbb{1}_{[0,t]}\otimes\phi,\mathbb{1}_{[0,t']}\otimes\phi')_{L^2_{t,k}}\\ &= (t\wedge t')\,(\phi,\phi')_{L^2_k},\end{aligned}$$

where $t \wedge t' = \min(t, t')$. The above means that $\widehat{W}$ behaves like Brownian motion in the time variable t and like white noise in the wave number variable k. The identity above is a rigorous version of the formal expression for the correlations

$$\mathbb{E}[\widehat{W}(t,k)\overline{\widehat{W}(t',k')}] = (t \wedge t')\,\delta(k-k').$$

Next let us introduce the linear operator $\widehat{A}$ given by

$$\begin{aligned} \widehat{A} : \widehat{f} \longmapsto (\widehat{A}\widehat{f})(k) &= c\,\mathrm{div}\left(\frac{k}{|k|}\,\widehat{f}(k)\right) + c\,\partial_{|k|}\log\big(\sigma(k)\big)\widehat{f}(k) - c\,\frac{d-1}{|k|}\,\widehat{f}(k) \\ &= c(\partial_{|k|}\widehat{f})(k) + cV(k)\widehat{f}(k) \end{aligned} \tag{2.2}$$

with domain

$$D(\widehat{A}) := \{\widehat{f} \in \widehat{\mathcal{H}} \,|\, \partial_{|k|}\widehat{f} \in \widehat{\mathcal{H}},\ V\,\widehat{f} \in \widehat{\mathcal{H}} \quad \text{and} \quad \widehat{f}_{|_{|k|=\kappa}} = 0\},$$

where $\widehat{f}_{|_{|k|=\kappa}}$ stands for the trace

$$\widehat{f}_{|_{|k|=\kappa}}(\kappa\theta) := \lim_{\varepsilon \to 0^+} \widehat{f}((\kappa+\varepsilon)\theta), \quad \forall \theta \in \mathbb{S}^{d-1}$$

which is well defined since $\widehat{f} \in \widehat{\mathcal{H}}$ and $\partial_{|k|}\widehat{f} \in \widehat{\mathcal{H}}$.

2.1.1. *Well-posedness in Fourier domain.*

Lemma 2.1. *Suppose that the function $k \mapsto S(k)$ satisfies Assumption 1.2. Then the operator $\widehat{A}$ generates a C^0-semigroup of $\widehat{\mathcal{H}}$ given by*

$$\big(\widehat{\Pi}_t\phi\big)(k) = \frac{\sigma\left(\frac{|k|-ct}{|k|}k\right)}{\sigma\,(k)}\,\phi\left(\frac{|k|-ct}{|k|}k\right)\mathbb{1}_{|k|-ct>\kappa}, \quad \forall\ k \in \mathbb{R}^d, \tag{2.3}$$

whose dual operator is given by

$$\big(\widehat{\Pi}_t^\star\phi\big)(k) = \left(\frac{|k|+ct}{|k|}\right)^{d-1}\,\frac{\sigma\,(k)}{\sigma\left(\frac{|k|+ct}{|k|}k\right)}\,\phi\left(\frac{|k|+ct}{|k|}k\right)\mathbb{1}_{|k|>\kappa}, \quad \forall\ k \in \mathbb{R}^d. \tag{2.4}$$

More precisely, one has for all $\phi, \phi' \in \widehat{\mathcal{H}}$

$$(\widehat{\Pi}_t\phi, \phi')_{L^2_k} = (\phi, \widehat{\Pi}_t^\star\phi')_{L^2_k}. \tag{2.5}$$

Proof. Our first goal is to show that for any $\widehat{\phi} \in \widehat{\mathcal{H}}$, the function $\widehat{\Pi}_t\widehat{\phi}$ defined by (2.3) is indeed an element of $\mathcal{H}$. It is straightforward to check that $\widehat{\Pi}_t\widehat{\phi}(k) = 0$ if $|k| < \kappa$. Applying the change of variable formula (1.11) with

$$F(k_1,k_2) = \left|\frac{\sigma(k_2)}{\sigma(k_1)}\,\widehat{\phi}(k_2)\right|^2,$$

one obtains

$$\begin{aligned} \int_{\mathbb{R}^d} \left|\widehat{\Pi}_t\widehat{\phi}(k)\right|^2 \mathrm{d}k &= \int_{\mathbb{R}^d} \left|\frac{\sigma\left(\frac{|k|-ct}{|k|}k\right)}{\sigma\,(k)}\,\widehat{\phi}\left(\frac{|k|-ct}{|k|}k\right)\right|^2 \mathbb{1}_{|k|-ct>\kappa}\,\mathrm{d}k \\ &= \int_{\mathbb{R}^d} \left(\frac{|k|+ct}{|k|}\right)^{d-1} \left(\frac{\sigma\,(k)}{\sigma\left(\frac{|k|+ct}{|k|}k\right)}\right)^2 |\widehat{\phi}(k)|^2 \mathbb{1}_{|k|>\kappa}\,\mathrm{d}k \end{aligned}$$

$$= \int_{\mathbb{R}^d} \frac{S\left(\frac{|k|+ct}{|k|}k\right)}{S(k)} |\widehat{\phi}(k)|^2 \, \mathrm{d}k,$$

and the desired result follows from (1.3) in Assumption 1.2.

It is straightforward to check $\left(\widehat{\Pi}_t\right)_{t\geq 0}$ is a C^0-semigroup, whose generator is given by

$$\partial_t[\widehat{\Pi}_t\widehat{\phi}] = -c\partial_{|k|}\widehat{\phi} - cV\widehat{\phi} = -\widehat{A}\widehat{\phi}, \text{for any } \widehat{\phi} \in D(\widehat{A}).$$

Finally, (2.5) follows from (1.11) with

$$F(k_1, k_2) = \frac{\sigma(k_2)}{\sigma(k_1)}\phi(k_2)\overline{\phi'(k_1)}, \qquad \phi, \phi' \in \widehat{\mathcal{H}}.$$

□

Lemma 2.1 guarantees that the initial value problem

$$\begin{cases} \partial_t \widehat{u}(t) + \widehat{A}\widehat{u}(t) = 0, \quad \forall\ t \geq 0, \\ \widehat{u}(0) = \widehat{u}_0, \end{cases} \tag{2.6}$$

is wellposed in $C^0([0,\infty), \widehat{\mathcal{H}})$, the space of space of time continuous function with value in the Hilbert space $\widehat{\mathcal{H}}$, for any initial condition $\widehat{u}_0 \in \widehat{\mathcal{H}}$, and that its solution is given by $\widehat{u}(t) = \widehat{\Pi}_t\widehat{u}_0$. This is by definition the mild solution of (2.6).

We are ready to introduce our stochastic model, given by the following stochastic evolution equation

$$\begin{cases} \mathrm{d}\widehat{U}(t) + \widehat{A}\,\widehat{U}(t)\,\mathrm{d}t = \widehat{\varphi}\,\mathrm{d}\widehat{W}(t), \quad \forall\ t \geq 0, \\ \widehat{U}(0) = 0. \end{cases} \tag{2.7}$$

A formal mild solution of the stochastic evolution equation is given by the expression

$$\widehat{U}(t) = \int_0^t \widehat{\Pi}_{t-s}\big(\widehat{\varphi}(\cdot)\,\mathrm{d}\widehat{W}(s)\big), \quad \forall\ t \geq 0, \tag{2.8}$$

which is interpreted in a weak sense: for any $\phi \in \widehat{\mathcal{H}}$, $\langle \widehat{U}(t), \phi\rangle$ is given by

$$\langle \widehat{U}(t), \phi\rangle = \int_0^t \langle \mathrm{d}\widehat{W}(s), \widehat{\varphi}(\cdot)\widehat{\Pi}^\star_{t-s}\phi\rangle, \quad \forall\ t \geq 0. \tag{2.9}$$

Note that the expression above is well-defined since $\widehat{\varphi}(\cdot)\widehat{\Pi}^\star_{t-s}\phi \in \widehat{\mathcal{H}}$ as consequence of $\widehat{\varphi} \in \widehat{\mathcal{H}} \cap L^\infty_k$ and the Hölder inequality.

2.1.2. *Correlation structure in Fourier domain.*

Proposition 2.2. *Suppose that φ and S satisfy Assumption 1.1 and Assumption 1.2, respectively. Then the process defined by* (2.8)-(2.9) *satisfies*

$$\mathbb{E}[\langle \widehat{U}(t), \phi\rangle\overline{\langle \widehat{U}(t), \phi'\rangle}] = \int_{\mathbb{R}^d} \Gamma(t,k)\phi(k)\overline{\phi'(k)}\,\mathrm{d}k, \qquad \forall \phi, \phi' \in \widehat{\mathcal{H}}, \tag{2.10}$$

where

$$\begin{aligned}\Gamma(t,k) &= \int_0^t \left|\widehat{\varphi}\left(\frac{|k|-cs}{|k|}k\right)\right|^2 \frac{S(k)}{S\left(\frac{|k|-cs}{|k|}k\right)} \mathbb{1}_{|k|-cs>\kappa}\,\mathrm{d}s \\ &= S(k)\int_{|k|-ct}^{|k|} \left|\widehat{\varphi}\left(s\frac{k}{|k|}\right)\right|^2 \frac{1}{S\left(s\frac{k}{|k|}\right)} \mathbb{1}_{s>\kappa}\frac{\mathrm{d}s}{c}.\end{aligned} \tag{2.11}$$

Moreover, $\sup_{t\geq 0}\Gamma(t,\cdot)\in L^1(\mathbb{R}^d)$.

Proof. From (2.9), one obtains

$$\mathbb{E}[\langle \widehat{U}(t),\phi\rangle\overline{\langle \widehat{U}(t),\phi'\rangle}] = \int_0^t \langle \widehat{\varphi}(\cdot)\widehat{\Pi}^\star_{t-s}\phi, \widehat{\varphi}(\cdot)\widehat{\Pi}^\star_{t-s}\phi'\rangle\,\mathrm{d}s. \tag{2.12}$$

Combining (2.12) and (2.4), and applying the change of variables formula (1.11) with

$$F(k_1,k_2) = |\widehat{\varphi}(k_2)|^2\left(\frac{|k_1|}{|k_2|}\right)^{d-1}\left(\frac{\sigma(k_2)}{\sigma(k_1)}\right)^2 \phi(k_1)\,\overline{\phi'(k_1)},$$

one obtains

$$\begin{aligned}\mathbb{E}[\langle \widehat{U}(t),\phi\rangle\overline{\langle \widehat{U}(t),\phi'\rangle}] &= \int_0^t \langle \widehat{\varphi}(\cdot)\widehat{\Pi}^\star_{t-s}\phi, \widehat{\varphi}(\cdot)\widehat{\Pi}^\star_{t-s}\phi'\rangle\,\mathrm{d}s \\ &= \int_0^t\int_{\mathbb{R}^d} |\widehat{\varphi}(k)|^2\left(\frac{|k|+cs}{|k|}\right)^{2(d-1)}\left(\frac{\sigma(k)}{\sigma\left(\frac{|k|+cs}{|k|}k\right)}\right)^2 \phi\left(\frac{|k|+cs}{|k|}k\right)\overline{\phi'\left(\frac{|k|+cs}{|k|}k\right)}\mathbb{1}_{|k|>\kappa}\,\mathrm{d}k\,\mathrm{d}s \\ &= \int_0^t\int_{\mathbb{R}^d} \left|\widehat{\varphi}\left(\frac{|k|-cs}{|k|}k\right)\right|^2\left(\frac{|k|}{|k|-cs}\right)^{d-1}\left(\frac{\sigma\left(\frac{|k|-cs}{|k|}k\right)}{\sigma(k)}\right)^2 \phi(k)\,\overline{\phi'(k)}\mathbb{1}_{|k|-cs>\kappa}\,\mathrm{d}k\,\mathrm{d}s \\ &= \int_0^t\int_{\mathbb{R}^d} \left|\widehat{\varphi}\left(\frac{|k|-cs}{|k|}k\right)\right|^2 \frac{S(k)}{S\left(\frac{|k|-cs}{|k|}k\right)}\phi(k)\,\overline{\phi'(k)}\mathbb{1}_{|k|-cs>\kappa}\,\mathrm{d}k\,\mathrm{d}s.\end{aligned}$$

Setting

$$\Gamma(t,k) = \int_0^t \left|\widehat{\varphi}\left(\frac{|k|-cs}{|k|}k\right)\right|^2 \frac{S(k)}{S\left(\frac{|k|-cs}{|k|}k\right)}\mathbb{1}_{|k|-cs>\kappa}\,\mathrm{d}s.$$

By Assumption 1.1,

$$\mathrm{supp}(\widehat{\varphi}) \subset \{k\in\mathbb{R}^d \mid \kappa\leq|k|\leq\kappa_\varphi\}.$$

Since S is continuous, cf. Assumption 1.2, there exists a positive constant $c(\kappa,\kappa_\varphi)$ such that

$$\inf_{0\leq s\leq \min\{t,\frac{|k|-\kappa}{c}\}} S\left(\frac{|k|-cs}{|k|}k\right)\mathbb{1}_{|k|>\kappa} \geq \inf_{\kappa\leq|k'|\leq\kappa_\varphi} S(k') := c(\kappa,\kappa_\varphi) > 0.$$

Define

$$\Phi\left(t,\frac{k}{|k|}\right) = \int_0^t \left|\widehat{\varphi}\left(s\,\frac{k}{|k|}\right)\right|^2\mathrm{d}s.$$

Since $\widehat{\varphi}$ is continous, the function $\Phi(t,\cdot):\mathbb{S}^{d-1}\to\mathbb{R}^+$ is continuous. As a result,

$$\Gamma(t,k) \leq \frac{S(k)}{c(\kappa,\kappa_\varphi)}\int_0^t \left|\widehat{\varphi}\left(s\,\frac{k}{|k|}\right)\right|^2\mathrm{d}s \leq \frac{S(k)}{c(\kappa,\kappa_\varphi)}\,\Phi\left(t,\frac{k}{|k|}\right) \lesssim \frac{S(k)}{c(\kappa,\kappa_\varphi)}\,\Phi\left(\kappa_\varphi,\frac{k}{|k|}\right).$$

Thanks to the continuity of $\Phi(\kappa_\varphi, \cdot)$, there exists some constant $C > 0$ such that $\Phi\left(\kappa_\varphi, \frac{k}{|k|}\right) \leq C$ for all $k \in \mathbb{R}^d$. In particular,

$$\sup_{t\geq 0} \Gamma(t,k) \leq \frac{C}{c(\kappa,\kappa_\varphi)}\, S(k),$$

which is in $L^1(\mathbb{R}^d)$ by Assumption 1.2.

The second line in (2.11) follows from the change of variables $s \mapsto |k| - cs$. □

2.2. **In the spatial domain.** In this section, we consider a physical space formulation of the model (2.7). One advantage of this point of view is that the mild solution may be viewed as a Gaussian random field, i.e. the random variable $U(t,x)$ will be well-defined for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^d$.

First, let us introduce a Wiener process $\big(W(t)\big)_{t\geq 0}$ defined as follows:

$$\langle W(t), \phi\rangle = \langle \xi, \mathbb{1}_{[0,t]} \otimes \phi\rangle, \qquad \text{for } t \in \mathbb{R}^+ \text{ and } \phi \in L^2_x. \tag{2.13}$$

Since the space-time white noise ξ is assumed to be real-valued, $W(t)$ is also real-valued and it suffices to deal with functions $\phi \in L^2_x$ which are-valued.

For any $t, t' \in \mathbb{R}^+$ and $\phi, \phi' \in L^2_x$, one has

$$\mathbb{E}[\langle W(t), \phi\rangle \langle W(t'), \phi'\rangle] = (t \wedge t')(\phi, \phi')_{L^2_x}.$$

Note that the Wiener process $\big(\widehat{W}(t)\big)_{t\geq 0}$ defined by (2.1) can be seen as the Fourier transform of the Wiener process $\big(W(t)\big)_{t\geq 0}$ defined by (2.13) above, indeed one has

$$\langle \widehat{W}(t), \phi\rangle = \langle W(t), \widehat{\phi}\rangle.$$

In the evolution equation (2.7) considered in the Fourier domain, the forcing is given by $\widehat{\varphi}(\cdot)\widehat{W}(t)$, therefore in the physical domain it should be given by the convolution $W^\varphi(t) = \varphi * W(t)$, which is defined as follows: for all $t \in \mathbb{R}^+$ and $\phi \in L^2(\mathbb{R}^d_x)$

$$\langle W^\varphi(t), \phi\rangle = \langle W(t), \varphi \star \phi\rangle, \tag{2.14}$$

where $\varphi \star \phi$ denotes the correlation product defined in Section 1.4. For any $t, t' \in \mathbb{R}^+$ and $\phi, \phi' \in L^2_x$, one has

$$\begin{aligned}
\mathbb{E}[\langle W^\varphi(t), \phi\rangle \langle W^\varphi(t'), \phi'\rangle] &= \mathbb{E}[\langle W(t), \varphi \star \phi\rangle \langle W(t'), \varphi \star \phi'\rangle] \\
&= (t \wedge t')(\varphi \star \phi, \varphi \star \phi')_{L^2_x} \\
&= (t \wedge t') \int_{\mathbb{R}^d_x} \int_{\mathbb{R}^d_x} C_\varphi(x - x')\phi(x)\phi(x')\,\mathrm{d}x\,\mathrm{d}x',
\end{aligned}$$

where the mapping C_φ is given by

$$C_\varphi(x) = \int_{\mathbb{R}^d_x} \varphi(x+y)\varphi(y)\,\mathrm{d}y = \big(\varphi \star \varphi\big)(x), \quad \forall\, x \in \mathbb{R}^d.$$

Note that the Fourier transform of C_φ is given by $\widehat{C}_\varphi(k) = |\widehat{\varphi(k)}|^2$ for all $k \in \mathbb{R}^d$.

One may interpret $\langle W^\varphi(t), \phi\rangle$ as the integral $\int_{\mathbb{R}^d_x} W^\varphi(t,x)\phi(x)\,\mathrm{d}x$, where $\big(W^\varphi(t,x)\big)_{t\geq 0, x\in\mathbb{R}^d}$ is a spatially homogeneous Gaussian random field which satisfies

$$\mathbb{E}[W^\varphi(t,x)W^\varphi(t',x')] = (t \wedge t')C_\varphi(x - x'), \quad \forall\, t, t' \in \mathbb{R}^+, \forall\, x, x' \in \mathbb{R}^d.$$

Next, let us introduce the operator A in physical space, corresponding to (2.2) in Fourier space, as follows:

$$A = \mathcal{F}^{-1}\widehat{A}\mathcal{F} \tag{2.15}$$

that is to say

$$Af = \mathcal{F}^{-1}[\widehat{A}\widehat{f}], \quad \forall f \in D(A) := \{f \in \mathcal{H} \,|\, \widehat{f} \in D(\widehat{A})\}.$$

It turns out that the operator A may be written as a pseudo-differential operator of the form:

$$Af(x) = \int_{\mathbb{R}^d_k} e^{2\pi i k\cdot x}\, a(x,k)\, \mathbb{1}_{|k|>\kappa}\, \widehat{f}(k)\, dk,$$

with symbol

$$a(x,k) = -c\,\frac{2\pi i\, k\cdot x + (d-1)}{|k|} + c\,V(k),$$

for regular function f.

Remark 2.3. *When $S(|k|) = |k|^\alpha$ for some $\alpha \in \mathbb{R}$, $V(k) = \alpha\,|k|^{-1}$ and the principal symbol of $a(x,k)$ is $-c\,\frac{2\pi i\, k\cdot x}{|k|}\,k$ which is a degree 0 symbol. See* [1, Proposition 4.6] *and the comments therein.*

Let us introduce the C^0-semigroups

$$\big(\Pi_t\big)_{t\geq 0} = \big(\mathcal{F}^{-1}\widehat{\Pi}_t\mathcal{F}\big)_{t\geq 0} \quad \text{and} \quad \big(\Pi_t^\star\big)_{t\geq 0} = \big(\mathcal{F}^{-1}\widehat{\Pi}_t^\star\mathcal{F}\big)_{t\geq 0} \tag{2.16}$$

of linear operators on $\mathcal{H}$. Lemma 2.1 shows that A generates the C^0-semigroup $\big(\Pi_t\big)_{t\geq 0}$ and $\big(\Pi_t^\star\big)_{t\geq 0}$ is the dual of $\big(\Pi_t\big)_{t\geq 0}$ in the sense that

$$(\Pi_t\phi, \phi')_{L^2_x} = (\phi, \Pi_t^\star\phi')_{L^2_x}, \tag{2.17}$$

for all $t \in \mathbb{R}^+$ and $\phi, \phi' \in \mathcal{H}$ as a consequence of the fact that the Fourier transform is an isometry from L^2_x to L^2_k.

2.2.1. *Wellposedness in space domain.* We are ready to introduce the initial value problem:

$$\begin{cases} \mathrm{d}U(t) + AU(t)\,\mathrm{d}t = \mathrm{d}W^\varphi(t), \quad \forall\, t \geq 0, \\ U(0) = 0. \end{cases} \tag{2.18}$$

which corresponds to (2.7) in physical space. In the spirit of the expression of the solution (2.8) in the Fourier domain, we claim that there exists a mild solution to (2.18) given by

$$U(t) = \int_0^t \Pi_{t-s}\,\mathrm{d}W^\varphi(s), \quad \forall\, t \in \mathbb{R}^+, \tag{2.19}$$

which as to be interpreted in a weak sense: for all $\phi \in \mathcal{H}$ one has

$$\langle U(t), \phi\rangle = \int_0^t \langle \mathrm{d}W^\varphi(s), \Pi_{t-s}^\star\phi\rangle \tag{2.20}$$

The definition above can be used to derive a more convenient expression for $\langle U(t), \phi\rangle$: one has with (1.8)

$$\langle U(t), \phi\rangle = \int_0^t \langle \mathrm{d}W(s), \varphi \star \big(\Pi_{t-s}^\star\phi\big)\rangle,$$

where the last stochastic integral above can be written as

$$\int_0^t \langle \mathrm{d}W(s), \varphi \star \big(\Pi_{t-s}^\star\phi\big)\rangle = \int_0^{+\infty} \langle \mathrm{d}W(s), \Phi_t(s,\cdot)\big)\rangle = \langle \xi, \Phi_t\rangle = \int_0^{+\infty}\int_{\mathbb{R}^d} \Phi_t(s,y)\,\mathrm{d}\xi(s,y),$$

with $\Phi_t(s,y) = \mathbb{1}_{[0,t]}(s)\big(\varphi \star \big(\Pi^\star_{t-s}\phi\big)\big)(y)$ for all $s \in \mathbb{R}^+$ and $y \in \mathbb{R}^d$. In particular, for all $s \in [0,t]$ and all $y \in \mathbb{R}^d$, one has

$$\Phi_t(s,y) = \int_{\mathbb{R}^d} \varphi(x+y)\big(\Pi^\star_{t-s}\phi\big)(x)\,\mathrm{d}x = (\varphi(\cdot+y), \Pi^\star_{t-s}\phi) = (\Pi_{t-s}\varphi(\cdot+y), \phi)$$

where the last equality follows by duality, cf. (2.17). We may therefore introduce the kernel G defined by:

$$G(t,x,y) = (\Pi_t\varphi(\cdot+y))(x), \quad \forall\ t \in \mathbb{R}^+, \forall\ x,y \in \mathbb{R}^d. \tag{2.21}$$

Combining the results above, one obtains

$$\langle U(t), \phi\rangle = \int_0^t \int_{\mathbb{R}^d} \langle G(t-s,\cdot,y), \phi\rangle\,\mathrm{d}\xi(s,y) = \langle \int_0^t \int_{\mathbb{R}^d} \langle G(t-s,\cdot,y)\,\mathrm{d}\xi(s,y), \phi\rangle.$$

The last equality is justified by the following result.

Proposition 2.4. *The function*

$$U(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s,x,y)\,\mathrm{d}\xi(s,y), \tag{2.22}$$

is a well-defined Gaussian random field which is a mild solution to (2.18) *in the sense of* (2.19)-(2.20)*. Moreover, for any multi-index β, any $t_1, t_2 > 0$ and any $x_1, x_2 \in \mathbb{R}^d$ there exists a (locally uniform) constant $C > 0$ such that*

$$\mathbb{E}|\partial_x^\beta U(t_1,x_1) - \partial_x^\beta U(t_2,x_2)|^2 \le C\left(|t_1-t_2| + |x_1-x_2|^2\right). \tag{2.23}$$

It remains to justify that the computations above are valid. Properties of the mapping G are given in Lemma 2.5 below. Moreover, one needs to show that the Gaussian random field $\big(U(t,x)\big)_{t\ge 0, x\in\mathbb{R}^d}$ is well-defined and to study its regularity properties.

Lemma 2.5. *Suppose that φ and S satisfy Assumption 1.1 and Assumption 1.2, respectively, and let $G(t,x,y)$ be defined by* (2.21)*. Then the following statements hold:*

(i) The Fourier transform in the x-variable of G is given by

$$\widetilde{G}(t,k,y) = e^{-2\pi i \frac{|k|-ct}{|k|}k\cdot y}\,\frac{\sigma\left(\frac{|k|-ct}{|k|}k\right)}{\sigma(k)}\varphi\left(\frac{|k|-ct}{|k|}k\right)\ \mathbb{1}_{|k|-ct>\kappa}\ . \tag{2.24}$$

(ii) The Fourier transform in the y-variable of G is given by

$$\widehat{G}(t,x,k) = e^{-2\pi i \frac{|k|+ct}{|k|}k\cdot x}\,\left(\frac{|k|+ct}{|k|}\right)^{d-1}\frac{\sigma(k)}{\sigma\left(\frac{|k|+ct}{|k|}k\right)}\,\overline{\widehat{\varphi}(k)}\,\mathbb{1}_{|k|>\kappa}. \tag{2.25}$$

(iii) For each multi-index β and for fixed $(t,x) \in \mathbb{R}^+\times\mathbb{R}^d$, $\partial_x^\beta G(t,x,\cdot) \in L^2(\mathbb{R}^d)$.

(iv) For all $y \in \mathbb{R}^d$, the mapping $G_y : (t,x) \in \mathbb{R}^+\times\mathbb{R}^d \mapsto G(t,x,y)$ is a solution to

$$\begin{cases} (\partial_t + A)G_y = 0, & \forall\ t > 0, \\ G(0,x,y) = \varphi(x+y), & \forall\ x,y \in \mathbb{R}^d\times\mathbb{R}^d. \end{cases}$$

Proof. (i) Using (2.3), (2.16) and (2.21), we obtain

$$G(t,x,y) = \int_{\mathbb{R}^d} e^{2\pi i\, k\cdot x}\, e^{-2\pi i \frac{|k|-ct}{|k|}k\cdot y}\, \frac{\sigma\left(\frac{|k|-ct}{|k|}k\right)}{\sigma(k)} \varphi\left(\frac{|k|-ct}{|k|}k\right)\ \mathbb{1}_{|k|-ct>\kappa}\ \mathrm{d}k, \tag{2.26}$$

from which the Fourier transform in the x-variable follows.

(ii) In order to compute the Fourier transform of G in the y variable, it suffices to perform the change of variables (1.11) on the right-hand side of (2.26). More precisely,

$$G(t,x,y) = \int_{\mathbb{R}^d_k} e^{2\pi i k\cdot y}\, e^{2\pi i \frac{|k|+ct}{|k|}k\cdot x}\, \left(\frac{|k|+ct}{|k|}\right)^{d-1}\ \frac{\sigma(k)}{\sigma\left(\frac{|k|+ct}{|k|}k\right)}\, \widehat{\varphi}(k)\ \mathbb{1}_{|k|>\kappa}\ \mathrm{d}k,$$

from which $\widehat{G}(t,x,k)$ follows.

(iii) By (1.3), there exists a constant $C(t)>0$ such that

$$\sup_{k\in\mathbb{R}^d} \frac{S\left(\frac{|k|+ct}{|k|}k\right)}{S(k)} \le C(t).$$

By the continuity of S, cf. Assumption 1.2, the function $t\mapsto C(t)$ is also continuous.

Differentiating (2.25) and using the Plancherel theorem, we obtain

$$\begin{aligned}
\left\|\partial_x^\beta G(t,x,\cdot)\right\|^2_{L^2(\mathbb{R}^d_y)} &= \left\|\partial_x^\beta \widehat{G}(t,x,\cdot)\right\|^2_{L^2(\mathbb{R}^d_k)} \qquad (2.27)\\
&\le \int_{\mathbb{R}^d_k} \left(\frac{|k|+ct}{|k|}\right)^{2(d-1)+2|\beta|} \frac{\sigma(k)^2}{\sigma\left(\frac{|k|+ct}{|k|}k\right)^2}\, |\widehat{\varphi}(k)|^2\, \mathbb{1}_{|k|>\kappa}\ \mathrm{d}k\\
&\le \int_{\mathbb{R}^d_k} \left(\frac{|k|+ct}{|k|}\right)^{(d-1)+2|\beta|} \frac{S\left(\frac{|k|+ct}{|k|}k\right)}{S(k)}\, |\widehat{\varphi}(k)|^2\, \mathbb{1}_{|k|>\kappa}\ \mathrm{d}k\\
&\le C(t)\, \left(1+\frac{ct}{\kappa}\right)^{d-1+2|\beta|} \|\varphi\|^2_{L^2} \qquad (2.28)
\end{aligned}$$

(iv) Note that

$$\left(\partial_t + c\,\partial_{|k|}\right)\left[(|k|-ct)\frac{k}{|k|}\right] = 0.$$

Applying the operator $\partial_t + c\,\partial_{|k|}$ to both sides of (2.24), it easily follows that:

$$\begin{aligned}
&\left(\partial_t + c\,\partial_{|k|}\right)\widetilde{G}(t,k,y)\\
&\quad= e^{-2\pi i \frac{|k|-ct}{|k|}k\cdot y}\, \sigma\left(\frac{|k|-ct}{|k|}k\right)\, \varphi\left(\frac{|k|-ct}{|k|}k\right)\ \mathbb{1}_{|k|-ct>\kappa} \left(\partial_t + c\,\partial_{|k|}\right)\left[\sigma(k)^{-1}\right]\\
&\quad= e^{-2\pi i \frac{|k|-ct}{|k|}k\cdot y}\, \sigma\left(\frac{|k|-ct}{|k|}k\right)\, \varphi\left(\frac{|k|-ct}{|k|}k\right)\ \mathbb{1}_{|k|-ct>\kappa}\, c\,\partial_{|k|}\left[\sigma(k)^{-1}\right]\\
&\quad= \widetilde{G}(t,k,y)\, c\,\sigma(k)\,\partial_{|k|}\left[\sigma(k)^{-1}\right].
\end{aligned}$$

Finally, it suffices to use the observation that

$$\sigma(k)\,\partial_{|k|}\left[\sigma(k)^{-1}\right] = -\sigma(k)^{-1}\,\partial_{|k|}\sigma(k) = -\partial_{|k|}\left(\log\sigma(k)\right) = -V(k).$$

□

Proof of Proposition 2.4. The fact that $U(t,x)$ in (2.22) is a well-defined Gaussian random variable follows from point (iii) in Lemma 2.5 and the Itô formula. Similarly, the proof that $U(t,x)$ solves (2.18) is a straight-forward application of (v) in Lemma 2.5. Finally, property (2.23) follows from the representation (2.22) and Lemma 2.5; the proof is similar to that of [1, Corollary 4.9] and we omit it. □

Remark 2.6. *Introducing an appropriate notion of weak solutions to* (2.18)*, one may prove that weak solutions are unique, which implies that* (2.22) *is the unique solution to* (2.18) *in this class. See Subsection 4.1.1. in* [1] *for additional details.*

2.2.2. *Correlation structure in space domain.* By Proposition 2.4, $U(t,x)$ in (2.22) is a well-defined Gaussian random field. Next we compute its two-point correlation.

Proposition 2.7. *Suppose that φ and S satisfy Assumption 1.1 and Assumption 1.2, respectively. Then the solution $U(t,x)$ to* (2.18) *has the following correlation structure*

$$\mathbb{E}[U(t,x)U(t,x')] = \int_{\mathbb{R}^d} e^{2\pi i k\cdot(x-x')}\Gamma(t,k)\,\mathrm{d}k, \tag{2.29}$$

where $\Gamma(t,k)$, given by (2.11)*, is absolutely integrable.*

Proof. Using the expression (2.22), the Plancherel theorem and (2.25), one obtains

$$\begin{aligned}
\mathbb{E}[U(t,x)U(t,x')] &= \int_0^t \int_{\mathbb{R}^d} G(t-s,x,y)G(t-s,x',y)\,\mathrm{d}s\,\mathrm{d}y \\
&= \int_0^t \int_{\mathbb{R}^d} \widehat{G}(t-s,x,k)\overline{\widehat{G}(t-s,x',k)}\,\mathrm{d}s\,\mathrm{d}k \\
&= \int_0^t \int_{\mathbb{R}^d} e^{2\pi i \frac{|k|+ct}{|k|}k\cdot(x-x')} \left(\frac{|k|+ct}{|k|}\right)^{2(d-1)} \frac{|\sigma(k)|^2}{\left|\sigma\left(\frac{|k|+ct}{|k|}k\right)\right|^2} |\widehat{\varphi}(k)|^2 \mathbb{1}_{|k|>\kappa}\,\mathrm{d}k\,\mathrm{d}s.
\end{aligned}$$

Using the change of variable formula (1.11), one then obtains

$$\begin{aligned}
&\mathbb{E}[U(t,x)U(t,x')] \\
&\quad= \int_0^t \int_{\mathbb{R}^d} e^{2\pi i k\cdot(x-x')} \left|\widehat{\varphi}\left(\frac{|k|-cs}{|k|}k\right)\right|^2 \left(\frac{|k|}{|k|-cs}\right)^{d-1} \left(\frac{\sigma\left(\frac{|k|-cs}{|k|}k\right)}{\sigma(k)}\right)^2 \mathbb{1}_{|k|-cs>\kappa}\,\mathrm{d}k\,\mathrm{d}s \\
&\quad= \int_0^t \int_{\mathbb{R}^d} e^{2\pi i k\cdot(x-x')} \left|\widehat{\varphi}\left(\frac{|k|-cs}{|k|}k\right)\right|^2 \frac{S(k)}{S\left(\frac{|k|-cs}{|k|}k\right)} \mathbb{1}_{|k|-cs>\kappa}\,\mathrm{d}k\,\mathrm{d}s \\
&\quad= \int_{\mathbb{R}^d} e^{2\pi i k\cdot(x-x')}\Gamma(t,k)\,\mathrm{d}k,
\end{aligned}$$

where we use formula (2.11) in the last step. The use of the Fubini theorem is allowed since $\Gamma(t,\cdot) \in L^1(\mathbb{R}^d)$ by Proposition 2.2. □

Note that the above result coincides with (2.10). Indeed, for all $\phi, \phi' \in \widehat{\mathcal{H}}$, (2.29) yields

$$\begin{aligned}\mathbb{E}[\langle \widehat{U}(t), \phi\rangle \overline{\langle \widehat{U}(t), \phi'\rangle}] &= \mathbb{E}[\langle U(t), \widehat{\phi}\rangle \overline{\langle U(t), \widehat{\phi'}\rangle}] \\ &= \mathbb{E}\left[\int_{\mathbb{R}^d_x \times \mathbb{R}^d_{x'}} \widehat{\phi}(x)\overline{\widehat{\phi'}(x')}\, U(t,x)\, \overline{U(t,x')}\, \mathrm{d}x\, \mathrm{d}x'\right] \\ &= \int_{\mathbb{R}^d_x \times \mathbb{R}^d_{x'}} \widehat{\phi}(x)\overline{\widehat{\phi'}(x')} \int_{\mathbb{R}^d_k} e^{2\pi i k\cdot(x-x')}\Gamma(t,k)\, \mathrm{d}k\, \mathrm{d}x\, \mathrm{d}x' \\ &= \int_{\mathbb{R}^d_k} \phi(k)\overline{\phi'(k)}\, \Gamma(t,k)\, \mathrm{d}k.\end{aligned}$$

The two formulations of the problem studied above are thus consistent.

2.3. **Asymptotic behavior.**

2.3.1. *Limiting correlation structure.* Under Assumption 1.1 and Assumption 1.2, one may use the dominated convergence theorem on the expression (2.11) to take the limit as $t \to \infty$, which yields:

$$\Gamma(\infty, k) := \lim_{t\to\infty} \Gamma(t,k) = \mathbb{1}_{|k|>\kappa}\, S(k) \int_\kappa^{|k|} \left|\widehat{\varphi}\left(s\frac{k}{|k|}\right)\right|^2 \frac{1}{S\left(s\frac{k}{|k|}\right)} \frac{\mathrm{d}s}{c}. \tag{2.30}$$

If $\widehat{\varphi}$ and S are radial, one has

$$\frac{\Gamma(\infty,k)}{S(k)} = \int_\kappa^r \left|\widehat{\varphi}\left(s\frac{k}{|k|}\right)\right|^2 \frac{1}{S\left(s\frac{k}{|k|}\right)} \frac{\mathrm{d}s}{c} = \mathfrak{C} \qquad \text{for all } |k| \geq \kappa_\varphi. \tag{2.31}$$

As a result, the ratio $\frac{\Gamma(\infty,k)}{S(k)}$ is constant for all $|k| \geq \kappa_\varphi$.

Remark 2.8. *For the purpose of numerical simulations, we will normalize $\widehat{\varphi}$ in such a way that the ratio $\frac{\Gamma(\infty,k)}{S(k)} = 1$ for large $|k|$, cf.* (4.9).

Corollary 2.9. *Suppose that φ and S satisfy Assumption 1.1 and Assumption 1.2, respectively. Then the solution $U(t,x)$ to* (2.18) *has the following asymptotic correlation structure*

$$\lim_{t\to\infty} \mathbb{E}[U(t,x)U(t,x')] = \int_{\mathbb{R}^d} e^{2\pi i k\cdot(x-x')} S(k)\, \vartheta(k)\, \mathrm{d}k, \tag{2.32}$$

where ϑ was defined in (1.7).

Proof. By Proposition 2.7,

$$\mathbb{E}[U(t,x)U(t,x')] = \int_{\mathbb{R}^d} e^{2\pi i k\cdot(x-x')}\Gamma(t,k)\, \mathrm{d}k,$$

where $\Gamma(t,k)$ is given in (2.11).

By Proposition 2.2, $\sup_{t\geq 0}\Gamma(t,\cdot) \in L^1(\mathbb{R}^d)$ and we may thus use the dominated convergence to take the limit $t \to \infty$ inside the integral, namely

$$\lim_{t\to\infty} \mathbb{E}[U(t,x)U(t,x')] = \int_{\mathbb{R}^d} e^{2\pi i k\cdot(x-x')}\, \Gamma(\infty,k)\, \mathrm{d}k.$$

Finally, by (2.30),

$$\Gamma(\infty,k) = S(k)\, \vartheta(k)$$

for ϑ as in (1.7). □

2.3.2. *Loss of regularity.* In the case $S(k) = |k|^{-(d+2H)}$, our next result shows that the energy cascade results in a *loss of regularity* phenomenon at infinite time.

Proposition 2.10. *Suppose that $S(k) = |k|^{-(d+2H)}$ for $H \in (0,1)$ and suppose that φ satisfies Assumption 1.1. Then the solution $U(t,x)$ to (2.18) converges in law to zero-mean Gaussian field $U_\infty(x)$ with correlation structure given by*

$$\mathbb{E}[U_\infty(x)\, U_\infty(x')] = \int_{\mathbb{R}^d_k} e^{2\pi i\, k\cdot(x-x')} |k|^{-d-2H}\, \vartheta(k)\, \mathrm{d}k \tag{2.33}$$

where

$$\vartheta(k) = \mathbb{1}_{|k|>\kappa} \int_\kappa^{|k|} s^{d+2H} \left| \widehat{\varphi}\left(s\frac{k}{|k|} \right) \right|^2 \mathrm{d}s. \tag{2.34}$$

Moreover, a loss of regularity *phenomenon takes place: while $x \mapsto U(t,x)$ is smooth for each finite time $t \geq 0$, $x \mapsto U_\infty(x)$ is only α-Hölder continuous for any $0 < \alpha < H$, namely*

$$\mathbb{E}|U_\infty(x) - U_\infty(x')|^2 \leq \frac{C}{H(1-H)} |x-x'|^{2H} + C\,|x-x'|^2, \quad \forall x, x' \tag{2.35}$$

where the constant $C > 0$ depends of H and d without blowing up when H tends to 0 or 1.

Proof. Formulas (2.33) and (2.34) follow from Corollary 2.9 since $S(k) = |k|^{-d-2H}$ satisfies Assumption 1.2. Next we set

$$V(t,x) = \int_0^t \int_{\mathbb{R}^d} G(s,x,y)\, \mathrm{d}\xi(s,y),$$

where G was defined in (2.21). Note that $V(t,x)$ has the same distribution as $U(t,x)$ in (2.22), as they are both Gaussian fields whose mean and correlations coincide by construction. Moreover, the Itô formula quickly shows that $V(t,x)$ converges to

$$U_\infty(x) = \int_0^\infty \int_{\mathbb{R}^d} G(s,x,y)\, \mathrm{d}\xi(s,y), \tag{2.36}$$

in $L^2(\Omega;\mathbb{C})$, since

$$\mathbb{E}|V(t,x) - U_\infty(x)|^2 = \frac{1}{2cH} \int_{\mathbb{R}^d_k} (|k| + ct)^{-2H}\, |k|^{2H+1}\, |\widehat{\varphi}(k)|^2\, \mathrm{d}k.$$

See [1, Corollary 4.11] for additional details.

Finally, using (2.32) together with the fact that $\mathbb{E}[U(t,x)\overline{U(t,x')}] = \mathbb{E}[V(t,x)\overline{V(t,x')}]$, the field (2.36) is shown to be a zero-mean Gaussian field with the desired correlation structure.

The proof of (2.35) may be found in [1, Corollary 4.12]. □

Remark 2.11. *It is easy to show that the inequality (2.35) is sharp as $|x-x'| \to 0$ by choosing $\widehat{\varphi}$ to be a (possibly regularized) indicator function over an isotropic region.*

Indeed, setting $\ell = x - x'$, one has

$$\mathbb{E}|U_\infty(x) - U_\infty(x')|^2 \geq \int_{\mathbb{R}^d_k} [1 - \cos(2\pi k\cdot\ell)]\, |k|^{-d-2H}\, \vartheta(k)\, \mathrm{d}k.$$

Assuming that $\widehat{\varphi}(k) = \mathbb{1}_{[1,2]}(|k|)$, we have that

$$|k|^{-2H-d}\, \vartheta(k) \geq |k|^{-2H-d}\, \mathbb{1}_{|k|>2}\, (2^{2H+d+1} - 1^{2H+d+1}).$$

As a result, for any $\ell \in \mathbb{R}^d$ with $|\ell| < 1$,

$$\mathbb{E}|U_\infty(x) - U_\infty(x')|^2 \gtrsim_H \int_0^\infty \int_{\mathbb{S}^{d-1}} \left[1 - \cos\left(2\pi|k||\ell|\,\gamma \cdot \frac{\ell}{|\ell|}\right)\right] \mathbb{1}_{|k|>2}\,|k|^{-2H-1}\,d\mu(\gamma)\;\mathrm{d}|k|$$
$$\gtrsim_H |\ell|^{2H} \int_0^\infty \int_{\mathbb{S}^{d-1}} \left[1 - \cos\left(2\pi r\,\gamma \cdot \frac{\ell}{|\ell|}\right)\right] \mathbb{1}_{r>2}\,r^{-2H-1}\,d\mu(\gamma)\;\mathrm{d}r \gtrsim_H |\ell|^{2H}$$

after changing variables to spherical coordinates so that $k \cdot \ell = |k|\,|\ell|\,\cos\theta$ and performing a dilation in the radial direction $r = |\ell|\,|k|$. Finally, one may repeat this argument with a slightly regularized $\mathbb{1}_{[1,2]}(|k|)$ in order to make $\widehat{\varphi}$ continuous.

3. Numerical method

In this section, we describe a numerical method which allows us to obtain numerical simulations for the model analyzed above. We explain below how the method introduced in the recent article [4] by the authors of this article can be generalized to deal with a general class of spectra which are not power laws.

To define the numerical method, it is more convenient to consider the formulation of the model in the Fourier domain.

3.1. Spatial discretization: finite volume approximation.

To define the numerical scheme, it is convenient to consider a conservative form of the first-order differential operator: one considers

(3.1)
$$\begin{cases} \mathrm{d}\widehat{U}(t,k) + c\,\nabla_k \cdot \left(\dfrac{k}{|k|}\widehat{U}(t,k)\right)\mathrm{d}t + c\widetilde{V}(k)\,\widehat{U}(t,k)\,\mathrm{d}t = \widehat{\varphi}(k)\,\mathrm{d}\widehat{W}(t,k), \quad \forall\, t>0, |k|>\kappa>0, \\ \widehat{U}(t,k) = 0, \quad \forall\, |k| \le \kappa, \\ \widehat{U}(0,k) = 0, \end{cases}$$

where the modified potential $\widetilde{V}$ is given by

$$\widetilde{V}(k) = V(k) - \frac{d-1}{|k|}.$$

Given a cell $\mathcal{K} \subset \mathbb{R}^d_k$, we denote by $|\mathcal{K}|$ its volume, by $\partial\mathcal{K}$ its boundary and by n the local unit outward normal vector to $\partial\mathcal{K}$. Due to the boundary conditions imposed in the evolution equation, all the cells are assumed to be subsets of $\{k \in \mathbb{R}^d \mid |k| \ge \kappa\}$.

To define a finite volume approximation, it is standard to consider averages of the solution over cells $\mathcal{K}$, i.e. to set

$$\widehat{U}_\mathcal{K}(t) = \frac{1}{|\mathcal{K}|}\int_\mathcal{K} \widehat{U}(t,k)\,\mathrm{d}k.$$

However the Gaussian random field $\widehat{U}$ is not defined pointwise. In this setting it is more convenient to consider

$$\widehat{U}_\mathcal{K}(t) = \langle \widehat{U}(t), \frac{1}{|\mathcal{K}|}\mathbb{1}_\mathcal{K}\rangle,$$

however for the first steps of the construction of the scheme let us make formal computations and assume that $\widehat{U}$ has a pointwise meaning.

Note that applying the Stokes formula, one obtains

$$\frac{1}{|\mathcal{K}|}\int_\mathcal{K} \nabla_k \cdot \left(\frac{k}{|k|}\widehat{U}(t,k)\right)\mathrm{d}k = \frac{1}{|\mathcal{K}|}\int_{\partial\mathcal{K}} \widehat{U}(t,k)\frac{k\cdot\mathrm{n}}{|k|}\,\mathrm{d}k.$$

The integration of the evolution equation over a cell $\mathcal{K}$ then gives

$$\mathrm{d}\widehat{U}_{\mathcal{K}}(t) + \frac{c}{|\mathcal{K}|}\int_{\partial\mathcal{K}} \widehat{U}(t,k)\frac{k\cdot \mathrm{n}}{|k|}\,\mathrm{d}k\,\mathrm{d}t + \frac{c}{|\mathcal{K}|}\int_{\mathcal{K}} \widehat{U}(t,k)\widetilde{V}(k)\,\mathrm{d}k\,\mathrm{d}t = \mathrm{d}\widehat{W}^{\varphi}_{\mathcal{K}}(t) \tag{3.2}$$

where $\widehat{W}^{\varphi}_{\mathcal{K}}(t)$ is defined as

$$\widehat{W}^{\varphi}_{\mathcal{K}}(t) = \langle \widehat{W}(t), \widehat{\varphi}\frac{1}{|\mathcal{K}|}\mathbb{1}_{\mathcal{K}}\rangle.$$

It is straightforward to check that $\big(\widehat{W}^{\varphi}_{\mathcal{K}}(t)\big)_{t\geq 0}$ is a complex-valued Wiener process, with

$$\mathbb{E}[\widehat{W}^{\varphi}_{\mathcal{K}}(t)\overline{\widehat{W}^{\varphi}_{\mathcal{K}}(t')}] = \frac{(t\wedge t')}{|\mathcal{K}|^2}\langle\widehat{\varphi}\mathbb{1}_{\mathcal{K}}, \widehat{\varphi}\mathbb{1}_{\mathcal{K}}\rangle = \frac{(t\wedge t')}{|\mathcal{K}|^2}\int_{\mathcal{K}} |\widehat{\varphi}(k)|^2\,\mathrm{d}k$$

Moreover, if $\mathcal{K}_1, \mathcal{K}_2$ are disjoint cells ($\mathcal{K}_1\cap\mathcal{K}_2 = \emptyset$), then one has

$$\mathbb{E}[\widehat{W}^{\varphi}_{\mathcal{K}_1}(t)\overline{\widehat{W}^{\varphi}_{\mathcal{K}_2}(t')}] = 0.$$

The evolution equation (3.2) for $\widehat{U}_{\mathcal{K}}(t)$ above is exact but cannot be implemented, two terms on the right-hand side need to be approximated. This requires to impose a choice for the cells.

As explained in [4], it is convenient to consider spherical coordinates, i.e. to write $k\neq 0$ as $k = r\theta$ where $r = |k| > 0$ is the radial component of k and $\theta = k/|k| \in \mathbb{S}^{d-1}$ is the angular component of k. Due to the boundary conditions in the model, one only need to consider a mesh of the set $\{k\in\mathbb{R}^d \mid |k|\geq\kappa\}$. The finite volume mesh is denoted by $\big(\mathcal{K}_{i,a}\big)_{i\geq 1, a\in\mathcal{A}}$, where $\mathcal{A}$ is a finite set, and the cell $\mathcal{K}_{i,a}$ is defined by

$$\mathcal{K}_{i,a} = \{k = |k|\theta \mid \rho_{i-\frac{1}{2}} < |k| < \rho_{i+\frac{1}{2}}; \theta\in\Theta_a\},$$

where $\big(\rho_{i-\frac{1}{2}}\big)_{i\geq 1}$ is an increasing sequence such that $\rho_{\frac{1}{2}} = \kappa$ and $\rho_{i-\frac{1}{2}} \underset{i\to+\infty}{\to} +\infty$, and where $\big(\Theta_a\big)_{a\in\mathcal{A}}$ is a mesh of the sphere $\mathcal{S}^{d-1}$. It is assumed that $\big(\Theta_a\big)_{a\in\mathcal{A}}$ is symmetric with respect to the origin, and that it is uniform. For instance, if $d=2$, this mesh can be defined by setting

$$\Theta_a = \{(\cos\vartheta, \sin\vartheta); a\Delta\vartheta < \vartheta < (a+1)\Delta\vartheta\},$$

for all $a\in\{0,\dots,2N_\vartheta - 1\}$, with angular resolution $\Delta\vartheta = \pi/N_\vartheta$.

The numerical scheme provides an approximation $\big(\widehat{U}_{i,a}\big)_{i\geq 1, a\in\mathcal{A}}$ of $\big(\widehat{U}_{\mathcal{K}_{i,a}}\big)_{i\geq 1, a\in\mathcal{A}}$. The main observation is that the boundary $\partial\mathcal{K}_{i,a}$ of a cell $\mathcal{K}_{i,a}$ defined as above can be decomposed into three parts, and that $k\cdot\mathrm{n}$ vanishes on one of these parts. Then using an upwind approximation, one can approximate the advection contribution as

$$\frac{c}{|\mathcal{K}_{i,a}|}\int_{\partial\mathcal{K}_{i,a}} \widehat{U}(t,k)\frac{k}{|k|}\cdot\mathrm{n}\,\mathrm{d}k \simeq c\frac{\widehat{U}_{i,a}(t) - \widehat{U}_{i-1,a}(t)}{h_i} + d_i\widehat{U}_{i,a}(t), \tag{3.3}$$

with the convention $\widehat{U}_{0,a}(t) = 0$, where h_i and d_i are defined for all $i\geq 1$ by

$$h_i = \frac{\rho^d_{i+\frac{1}{2}} - \rho^d_{i-\frac{1}{2}}}{\rho^{d-1}_{i-\frac{1}{2}}}, \quad d_i = cd\frac{\rho^{d-1}_{i+\frac{1}{2}} - \rho^{d-1}_{i-\frac{1}{2}}}{\rho^d_{i+\frac{1}{2}} - \rho^d_{i-\frac{1}{2}}}. \tag{3.4}$$

We refer to [4] for the details.

Concerning the last term that remains to be dealt with in (3.2) when $\mathcal{K} = \mathcal{K}_{i,a}$, one can approximate

$$\frac{1}{|\mathcal{K}_{i,a}|}\int_{\mathcal{K}_{i,a}} \widehat{U}(t,k)\widetilde{V}(k)\,\mathrm{d}k \approx \frac{\widehat{U}_{i,a}(t)}{|\mathcal{K}_{i,a}|}\int_{\mathcal{K}_{i,a}} \widetilde{V}(k)\,\mathrm{d}k.$$

For all $i \geq 1$, set

$$\rho_i = \frac{\rho_{i+\frac{1}{2}} + \rho_{i-\frac{1}{2}}}{2},$$

and for all $a \in \mathcal{A}$ let $\theta_a \in \Theta_a$ be the center of Θ_a. Recalling the definition of $\widetilde{V}$, one has

$$\begin{aligned}\frac{1}{|\mathcal{K}_{i,a}|}\int_{\mathcal{K}_{i,a}} \widetilde{V}(k)\,\mathrm{d}k &= \frac{1}{|\mathcal{K}_{i,a}|}\int_{\mathcal{K}_{i,a}} V(k)\,\mathrm{d}k - \frac{1}{|\mathcal{K}_{i,a}|}\int_{\mathcal{K}_{i,a}} \frac{d-1}{|k|}\,\mathrm{d}k \\ &= \frac{1}{|\mathcal{K}_{i,a}|}\int_{\mathcal{K}_{i,a}} V(k)\,\mathrm{d}k - d\frac{\rho_{i+\frac{1}{2}}^{d-1} - \rho_{i-\frac{1}{2}}^{d-1}}{\rho_{i+\frac{1}{2}}^{d} - \rho_{i-\frac{1}{2}}^{d}} \\ &\approx V(\rho_i\theta_a) - d_i.\end{aligned}$$

For all $i \geq 1$ and $a \in \mathcal{A}$, set

$$\widehat{W}_{i,a}^{\varphi}(t) = \widehat{W}_{\mathcal{K}_{i,a}}^{\varphi}(t), \quad \forall\, t \geq 0.$$

Combining the approximations above, the spatial discretization of the model is given by the system of stochastic differential equations

(3.5)
$$\begin{cases} \mathrm{d}\widehat{U}_{i,a}(t) + c\dfrac{\widehat{U}_{i,a}(t) - \widehat{U}_{i-1,a}(t)}{h_i}\,\mathrm{d}t + cV(\rho_i\theta_a)\widehat{U}_{i,a}(t)\,\mathrm{d}t = \mathrm{d}\widehat{W}_{i,a}^{\varphi}(t), \quad \forall\, t \geq 0,\ i \geq 1,\ a \in \mathcal{A}, \\ \widehat{U}_{0,a}(t) = 0, \quad \forall\, t \geq 0,\ a \in \mathcal{A}, \\ \widehat{U}_{i,a}(0) = 0, \quad \forall\, i \geq 1,\ a \in \mathcal{A}. \end{cases}$$

The cells $\left(\mathcal{K}_{i,a}\right)_{i\geq 1, a\in\mathcal{A}}$ in the finite volume mesh are pairwise disjoint. Moreover, each cell $\mathcal{K}_{i,a}$ has a symmetric cell with respect to 0 in the mesh, which can be denoted by $\mathcal{K}_{i,-a}$. If $i \neq j$, for all $a, b \in \mathcal{A}$, the Wiener processes $\widehat{W}_{i,a}^{\varphi}$ and $\widehat{W}_{j,b}^{\varphi}$ are independent. Moreover, for any $i \geq 1$, given $a, b \in \mathcal{A}$, if $b \notin \{a, -a\}$, the Wiener processes $\widehat{W}_{i,a}^{\varphi}$ and $\widehat{W}_{i,b}^{\varphi}$ are independent. Finally, $\overline{\widehat{W}_{i,a}^{\varphi}} = \widehat{W}_{i,-a}$ for all $i \geq 1$ and $a \in \mathcal{A}$. The properties above are summarized writing

$$\begin{cases} \mathbb{E}[\widehat{W}_{i,a}^{\varphi}(t)\overline{\widehat{W}_{j,b}^{\varphi}(s)}] = \delta_{i,j}\delta_{a,b}\dfrac{(t\wedge s)}{|\mathcal{K}_{i,a}|^2}\displaystyle\int_{\mathcal{K}_{i,a}} |\widehat{\varphi}(k)|^2\,\mathrm{d}k, \quad \forall\, t, s \geq 0,\ i, j \geq 1,\ a, b \in \mathcal{A}, \\ \overline{\widehat{W}_{i,a}^{\varphi}(t)} = \widehat{W}_{i,-a}(t), \quad \forall\, t \geq 0,\ i \geq 1,\ \in \mathcal{A}. \end{cases}$$

3.2. **Temporal discretization: splitting algorithm.** Let us now present the temporal discretization scheme applied to the semi-discrete approximation (3.5). The time-step size is denoted by Δt, and for all $n \geq 0$ set $t_n = n\Delta t$. Like in [4] we propose to use a Lie–Trotter splitting scheme, which on each interval $[t_n, t_{n+1}]$ combines solutions of two systems. First, one considers the system of Ornstein–Uhlenbeck stochastic differential equations

$$\mathrm{d}\widehat{U}_{i,a}^{\mathrm{ou}}(t) + cV(\rho_i\theta_a)\widehat{U}_{i,a}^{\mathrm{ou}}(t)\,\mathrm{d}t = \mathrm{d}\widehat{W}_{i,a}^{\varphi}(t), \quad \forall\, t \geq 0,\ i \geq 1,\ a \in \mathcal{A}. \tag{3.6}$$

Second, one considers the deterministic linear system of differential equations

$$\begin{cases} \partial_t\widehat{U}_{i,a}^{\mathrm{ad}}(t) + c\dfrac{\widehat{U}_{i,a}^{\mathrm{ad}}(t) - \widehat{U}_{i-1,a}^{\mathrm{ad}}(t)}{h_i} = 0, \quad \forall\, t \geq 0,\ i \geq 1,\ a \in \mathcal{A}, \\ \widehat{U}_{0,a}^{\mathrm{ad}}(t) = 0, \quad \forall\, t \geq 0,\ a \in \mathcal{A}, \end{cases} \tag{3.7}$$

which takes into account only the advection dynamics.

Numerical approximation for (3.6). Given $i, j \geq 1$ and $a, b \in \mathcal{A}$, the stochastic differential equations (3.6) for $\big(\widehat{U}^{\rm ou}_{i,a}(t)\big)_{t\geq 0}$ and $\big(\widehat{U}^{\rm ou}_{j,b}(t)\big)_{t\geq 0}$ are driven by independent Wiener processes $\widehat{W}^{\varphi}_{i,a}$ and $\widehat{W}^{\varphi}_{j,b}$, except if one has $i = j$ and $a \in \{-b, b\}$.

Given $i \geq 1$ and $a \in \mathcal{A}$, the solution of (3.6) is given by

$$\widehat{U}^{\rm ou}_{i,a}(t) = e^{-cV(\rho_i\theta_a)t}\widehat{U}^{\rm ou}_{i,a}(0) + \int_0^t e^{-cV(\rho_i\theta_a)(t-s)}\,\mathrm{d}\widehat{W}^{\varphi}_{i,a}(s).$$

The solution at time t_{n+1} given the solution at time t_n can be expressed as

$$\widehat{U}^{\rm ou}_{i,a}(t_{n+1}) = e^{-cV(\rho_i\theta_a)\Delta t}\widehat{U}^{\rm ou}_{i,a}(t_n) + \int_{t_n}^{t_{n+1}} e^{-cV(\rho_i\theta_a)(t_{n+1}-s)}\,\mathrm{d}\widehat{W}^{\varphi}_{i,a}(s).$$

Given $i \geq 1$ and $a \in \mathcal{A}$, the random variables $\int_{t_n}^{t_{n+1}} e^{-cV(\rho_i\theta_a)(t_{n+1}-s)}\,\mathrm{d}\widehat{W}^{\varphi}_{i,a}(s)$ indexed by $n \geq 0$ are independent complex-valued Gaussian random variables, which can be expressed as

$$\int_{t_n}^{t_{n+1}} e^{-cV(\rho_i\theta_a)(t_{n+1}-s)}\,\mathrm{d}\widehat{W}^{\varphi}_{i,a}(s) = \varrho_{i,a}\widehat{\gamma}^n_{i,a}$$

where

$$\varrho_i = \sqrt{\frac{1 - e^{-2\Delta t V(\rho_i\theta_a)}}{2V(\rho_i\theta_a)\,|\mathcal{K}_{i,a}|^2}\int_{\mathcal{K}_{i,a}} |\widehat{\varphi}(k)|^2\,\mathrm{d}k}, \quad \forall\ i \geq 1,\ a \in \mathcal{A}, \tag{3.8}$$

and $\big(\widehat{\gamma}^n_{i,a}\big)_{n\geq 0, i\geq 1, a\in\mathcal{A}}$ are Gaussian random variables that satisfy

$$\begin{cases} \mathbb{E}[\widehat{\gamma}^n_{i,a}\overline{\widehat{\gamma}^m_{j,b}}] = \delta_{i,j}\delta_{a,b}\delta_{n,m}, \quad \forall\ n, m \geq 0,\ i, j \geq 1,\ a, b \in \mathcal{A}, \\ \overline{\widehat{\gamma}^n_{i,a}} = \widehat{\gamma}^n_{i,-a}, \quad \forall\ n \geq 0,\ i \geq 1,\ \in \mathcal{A}. \end{cases}$$

If one defines

$$\begin{cases} \widehat{U}^{n+1,\rm ou}_{i,a} = e^{-cV(\rho_i\theta_a)\Delta t}\widehat{U}^{n,\rm ou}_{i,a} + \varrho_{i,a}\widehat{\gamma}^n_{i,a}, \quad \forall\ n \geq 0,\ i \geq 1,\ a \in \mathcal{A}, \\ \widehat{U}^{0,\rm ou}_{i,a} = \widehat{U}^{\rm ou}_{i,a}(0), \quad \forall\ i \geq 1,\ a \in \mathcal{A}, \end{cases}$$

then for any $n \geq 0$ the Gaussian random variables $\big(\widehat{U}^{n,\rm ou}_{i,a}\big)_{i\geq 1, a\in\mathcal{A}}$ and $\big(\widehat{U}^{\rm ou}_{i,a}(t_n)\big)_{i\geq 1, a\in\mathcal{A}}$ are equal in distribution.

Numerical approximation for (3.7). The system of differential equations (3.7) cannot be solved exactly. Applying the standard explicit Euler method, one obtains the numerical scheme

$$\begin{cases} \widehat{U}^{n+1,\rm ad}_{i,a} - \widehat{U}^{n,\rm ad}_{i,a} + \dfrac{c\Delta t}{h_i}\big(\widehat{U}^{n,\rm ad}_{i,a} - \widehat{U}^{n,\rm ad}_{i-1,a}\big) = 0, \quad \forall\ n \geq 0,\ i \geq 1,\ a \in \mathcal{A}, \\ \widehat{U}^{n,\rm ad}_{0,a} = 0, \quad \forall\ n \geq 0,\ a \in \mathcal{A}, \end{cases}$$

which can be interpreted as the fully discrete upwind scheme for linear advection equation in dimension 1

$$\begin{cases} \partial_t\widehat{U}^{\rm ad}_a(t, r) + c\partial_r\widehat{U}^{\rm ad}_a(t, r) = 0, \quad \forall\ t \geq 0,\ r \geq \kappa, \\ \widehat{U}(t, \kappa) = 0, \forall\ t \geq 0, \end{cases}$$

which describes the advection in the radial variable $r = |k|$, for any fixed $a \in \mathcal{A}$, on a mesh with cells of length h_i for $i \geq 1$. It is well-known that ensuring the stability of the above the scheme requires to impose the Courant–Friedrichs–Lewy stability condition

$$\frac{c\Delta t}{h_i} \leq 1, \quad \forall\ i \geq 1.$$

To avoid numerical dissipation and preserve the fundamental properties of the model at the discrete time level, we choose to impose the stronger condition

$$\frac{c\Delta t}{h_i} = 1, \quad \forall\, i \geq 1. \tag{3.9}$$

As a result, the sequence $\left(\rho_{i+\frac{1}{2}}\right)_{i\geq 0}$ is chosen such that $h_i = c\Delta t$ is independent of $i \geq 1$, depending on the choice of the time-step size Δt: owing to (3.4), this holds if the sequence is defined recursively by

$$\begin{cases} \rho_{i+\frac{1}{2}} = \left(\rho^d_{i-\frac{1}{2}} + c\Delta t \rho^{d-1}_{i-\frac{1}{2}}\right)^{\frac{1}{d}}, \quad i \geq 1, \\ \rho_{\frac{1}{2}} = \kappa. \end{cases} \tag{3.10}$$

Due to the condition (3.9), the solution of the fully discrete upwind scheme above is given by the simple formula

$$\widehat{U}^{n+1,\mathrm{ad}}_{i,a} = \widehat{U}^{n,\mathrm{ad}}_{i-1,a}, \quad \forall\, n \geq 0,\ i \geq 1,\ a \in \mathcal{A}.$$

Splitting scheme. The fully discrete scheme is constructed using a splitting method. Let $\left(\widehat{U}^n_{i,a}\right)_{i\geq 1, a\in\mathcal{A}}$ denote the numerical solution at iteration $n \geq 0$, which is meant to be an approximation of the solution $\left(\widehat{U}_{i,a}(t_n)\right)_{i\geq 1,a\in\mathcal{A}}$ at time t_n to the semi-discrete system (3.5). Given the numerical solution $\left(\widehat{U}^n_{i,a}\right)_{i\geq 1,a\in\mathcal{A}}$ at iteration n, the numerical solution $\left(\widehat{U}^{n+1}_{i,a}\right)_{i\geq 1,a\in\mathcal{A}}$ at iteration $n+1$ is defined by integrating first (3.6) and second (3.7), on the time interval $[t_n, t_{n+1}]$, using the numerical methods described above. We obtain the following numerical scheme: for all $n \geq 0$, $i \geq 1$ and $a \in \mathcal{A}$,

$$\begin{cases} \widehat{U}^{n+\frac{1}{2}}_{i,a} = e^{-cV(\rho_i\theta_a)\Delta t}\widehat{U}^n_{i,a} + \varrho_{i,a}\widehat{\gamma}^n_{i,a}, \\ \widehat{U}^{n+1}_{i,a} = \widehat{U}^{n+\frac{1}{2}}_{i-1,a}, \end{cases} \tag{3.11}$$

which is supplemented with the boundary conditions and the initial values

$$\begin{cases} \widehat{U}^n_{0,a} = 0, \quad \forall\, n \geq 0,\ a \in \mathcal{A}, \\ \widehat{U}^0_{i,a} = 0, \quad \forall\, i \geq 1,\ a \in \mathcal{A}. \end{cases} \tag{3.12}$$

4. Numerical simulations: application to the JONSWAP spectrum

The objective of this section is to illustrate the behavior of the model proposed numerical scheme (3.11)-(3.12) to generate the JONSWAP spectrum given by (1.1). Note that this spectrum is radial. In addition, $\widehat{\varphi}$ is chosen in the sequel to be radial, i.e. one has $\widehat{\varphi}(k) = \widehat{\varphi}(|k|)$.

Before presenting the numerical simulations, two important aspects are discussed: how to compare the numerical and theoretical spectra, and how to retrieve an approximation in the spatial domain from the finite volume approximation in the Fourier domain.

4.1. **Wave-number spectrum and physical space representation.**

The results of numerical simulations need to be compared to the theoretical predictions obtained for the continuous model and the expression of the spectrum. At any time $t \geq 0$, the power spectral density (PSD) is defined by

$$\Gamma(t,k) = \int_{\mathbb{R}^d} e^{-2i\pi k\cdot x}\mathbb{E}\left[U(t,0)U(t,x)\right]\,\mathrm{d}x. \tag{4.1}$$

When computing the second order moment of the Gaussian random variable $\widehat{U}_{\mathcal{K}}(t)$, which is the average of $\widehat{U}(t,k)$ over a finite volume cell $\mathcal{K}$, one obtains

$$\Gamma_{\mathcal{K}}(t) = \mathbb{E}\left|\widehat{U}_{\mathcal{K}}(t)\right|^2 = \frac{1}{|\mathcal{K}|^2}\int_{\mathcal{K}} \Gamma(t,k)\,\mathrm{d}k. \tag{4.2}$$

From (4.2), to compute the average of the PSD $\Gamma(t,k)$ over a finite volume cell $\mathcal{K}$, one needs to consider $|\mathcal{K}|\,\Gamma_{\mathcal{K}}(t)$, which is therefore the appropriate quantity employed below when comparing the numerical and theoretical spectra. The additional factor $|\mathcal{K}|$ is due to the distributional nature of the field $\widehat{U}$ which is delta correlated and thus of infinite variance. In addition, since the JONSWAP spectrum depends on the radial variable only, we consider the angle averaged version of the spectrum, which is defined by

$$\Gamma_i^{\Theta}(t) = \frac{1}{|\mathcal{A}|}\sum_{a\in\mathcal{A}} \Gamma_{\mathcal{K}_{i,a}}(t). \tag{4.3}$$

It is worth mentioning that the proposed finite volume discretization in the Fourier domain is not associated with a natural method to provide a numerical field defined in the spatial domain, since the finite volume mesh is not a standard Cartesian mesh. To obtain a representation of the numerical solution in the physical space, we propose the field defined by

$$\tilde{U}(t,x) = \sum_{n=1}^{N}\sum_{a\in\mathcal{A}} e^{2i\pi k_{na}\cdot x}\widehat{U}_{\mathcal{K}_{n,a}}(t)\rho_n^{d-1}\Delta\rho_n\Delta\Theta_a, \quad \forall\, x\in\mathbb{R}^d. \tag{4.4}$$

In the expression above, N is the numerical resolution in the radial direction, $\mathcal{A}$ is the set indexing the finite volume discretization of the unit $(d-1)-$dimensional sphere and $\Delta\Theta_a$ is the corresponding differential solid angle at the angular coordinate a. The field constructed this way is statistically homogeneous. In addition, it satisfies a discrete isotropy property: its correlation function is invariant under the discrete set of rotations that preserve the mesh $(\Theta_a)_{a\in\mathcal{A}}$. Numerically, the variable $x\in\mathbb{R}^d$ will be discretized on a Cartesian box of size L with step Δx in every direction. In practice, we choose the mesh in physical space such that $L\simeq\frac{1}{\kappa}$ and $\Delta x\le\rho_N^{-1}$. The discretized version of the continuous field $\widetilde{U}$ will be denoted $\widetilde{U}_\Delta$.

4.2. **Numerical results.** We are now going to provide numerical illustration in dimension $d=2$ of the model for the JONSWAP spectrum

$$S_{\mathrm{J}}(|k|) = \alpha g^{-\frac{1}{2}}|k|^{-5/2}\exp\left(-\frac{5}{4}\left(\frac{k_p}{|k|}\right)^2\right)\gamma^{\exp\left[-\frac{1}{2\sigma^2}\left(\sqrt{\frac{|k|}{k_p}}-1\right)^2\right]}, \tag{4.5}$$

given by (1.2) with the dispersion relation $\omega(k)=\sqrt{g|k|}$. The potential function V defined by (1.5) is given by

$$V(|k|) = \frac{1}{2|k|}\left(d+\frac{3}{2}-\frac{5}{2}\left(\frac{k_p}{|k|}\right)^2\right) + \frac{\ln\gamma}{4\sigma^2}\frac{\exp\left(-\frac{1}{2\sigma^2}\left(\sqrt{\frac{|k|}{kp}}-1\right)^2\right)}{\sqrt{|k|/k_p}}\left(\sqrt{\frac{|k|}{k_p}}-1\right). \tag{4.6}$$

We will also consider Pierson-Moskowitz spectrum that will be denoted by S_{PM} obtained by choosing $\gamma=1$ in (4.5). Owing to (2.30), in the large time regime, one has

$$\Gamma(\infty,|k|) = \lim_{t\to+\infty}\Gamma(t,k) = \mathbb{1}_{|k|>\kappa}\,S_{\mathrm{J}}(|k|)\int_{\kappa}^{|k|}\left|\widehat{\varphi}\,(s)\right|^2\frac{1}{S_{\mathrm{J}}\,(s)}\frac{\mathrm{d}s}{c}. \tag{4.7}$$

At this point, we have to take care of two points to compare $S_{\mathrm{J}}(|k|)$ and $\Gamma(\infty,|k|)$. First of all, the forcing $\widehat{\varphi}$ is compactly supported in Fourier space. Therefore, the integral on the right hand

N	N_θ	h	κ	c	α	γ	k_p	g	σ^2	ρ_1	ρ_2
2^{13}	2^8	10^{-2}	$1/2$	1	1	3.3(1)	5	9.81	0.09	$k_p/2$	ρ_1+1

TABLE 1. Simulations parameters. For γ, the two values correspond to the JONSWAP and Pierson-Moskowitz spectra.

side of (4.7) depends on $|k|$ but is constant for large enough $|k|$. Secondly, it is clear from (4.7) that multiplying $S_{\mathrm{J}}(|k|)$ by a multiplicative constant does not modify $\Gamma(\infty,|k|)$. The value of a multiplicative constant in $\Gamma(\infty,|k|)$ can be settled by the choice of the forcing $\widehat{\varphi}$. In the following, the forcing term will be based on a bump function $\widehat{\psi}$ given by

$$\widehat{\psi}(k) = \begin{cases} \exp\left[-\dfrac{1}{(|k|-\rho_1)(\rho_2-|k|)}\right], \text{ if } \quad \forall \rho_1 < |k| < \rho_2, \\ 0 \text{ else.} \end{cases} \tag{4.8}$$

The forcing $\widehat{\varphi}$ is then chosen such that,

$$\widehat{C}_\varphi(|k|) = |\widehat{\varphi}(|k|)|^2 = \mathcal{N}\widehat{\psi}(|k|)^2, \quad \mathcal{N} = \frac{c}{\displaystyle\int_\kappa^{+\infty} S_{\mathrm{J}}(s)^{-1}\widehat{\psi}^2(s)\,\mathrm{d}s}. \tag{4.9}$$

By doing so, the PSD (4.7) satisfies

$$\Gamma(\infty,|k|) = S_{\mathrm{J}}(|k|)I(|k|), \quad \forall\ |k| \geq \kappa,$$

where the auxiliary mappping I satisfies

$$I(|k|) = \frac{\displaystyle\int_\kappa^{|k|} S_{\mathrm{J}}(s)^{-1}\widehat{\psi}(s)^2\,\mathrm{d}s}{\displaystyle\int_\kappa^{+\infty} S_{\mathrm{J}}(s)^{-1}\widehat{\psi}(s)^2\,\mathrm{d}s} = \begin{cases} 1, \text{ if } \quad |k| \geq \rho_2, \\ 0, \text{ if } \quad |k| \leq \rho_1. \end{cases} \tag{4.10}$$

Through this mapping, and therefore the forcing support, one controls the regions where the model coincides with the desired spectrum. The simulations presented below are run with $\widehat{C_f}$ given by (4.9) and V by (4.6). The parameters used in the simulations for the JONSWAP and Pierson-Moskowitz spectra are given in Table 1 along with the parameters of the numerical approximation procedure.

Let us now present the results of numerical simulations performed with these parameters.

In Figure 1, we present the results of numerical simulations conducted with the set of parameters given in Table 1. The left panel of the figure compares the expected spectrum S_{J} (dotted lines) with the estimation of the angle averaged power spectral densities (solid lines) in the JONSWAP and Pierson-Moskowitz cases. In order to have a statistical estimation of these spectra, we first integrate the dynamics until the time $t^* = \rho_N/c$ is reached. For times larger than t^*, we consider that we are in a statistically steady state. Once in the statistically steady state, the PSD are estimated by averaging $M = 500$ realizations of the squared modulus of the finite volume field every $t^*/5$ units of time. By doing doing so we consider that waiting t^* units of time is sufficient for the instances of the field to be independent – which is not the case but is convenient numerically. The left panel of Figure 1 shows a good agreement between the estimated spectra obtained numerically and the expected ones. In particular, for $\rho \geq \rho_2$ the spectra coincide with the JONSWAP and Pierson-Moskowitz one while it vanishes for $\rho \leq \rho_1$ and is different from S_{J} for $\rho_1 < \rho < \rho_2$. The shape of the spectra obtained by the numerical method for $\rho_1 < \rho < \rho_2$ is settled by the forcing.

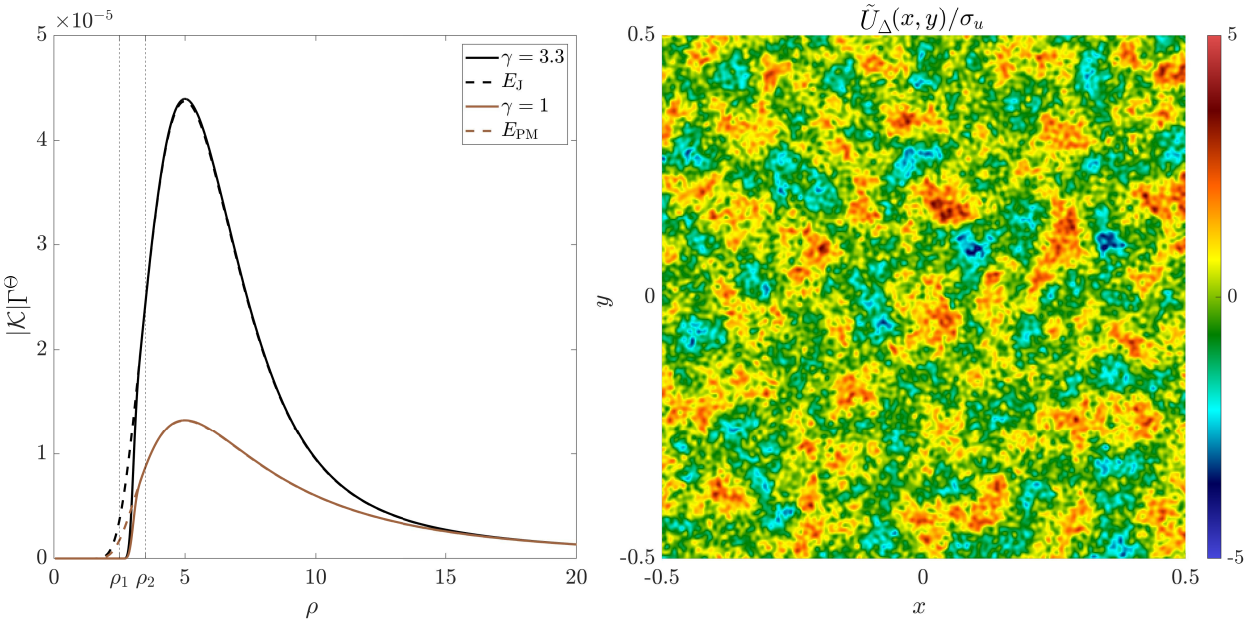

FIGURE 1. Left, Angle averaged spectra obtained numerically (solid lines) compared to the JONSWAP and Pierson-Mosowitz spectra (4.5) (dotted lines). The black lines correspond to the JONSWAP ($\gamma = 3.3$) case while the brown lines are the Pierson-Moskowitz ($\gamma = 1$) case. The vertical dotted lines corresponds to the radii ρ_1 and ρ_2 entering the definition of the forcing term. Right, physical space representation $\widetilde{U}_\Delta$ of the JONSWAP field at a given time in the statistically steady state. σ_u is the expected standard deviation for the solution of the dynamics.

On the right panel of Figure 1, we show the physical space representation of a snapshot of the Fourier space field. This field is obtained using (4.4) in dimension $d = 2$. The displayed field indeed exhibit statistical homogeneity and isotropy.

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