Cours ENSL:
Big Data – Streaming, Sketching, Compression

Olivier Beaumont, Inria Bordeaux Sud-Ouest
Olivier.Beaumont@inria.fr

Introduction

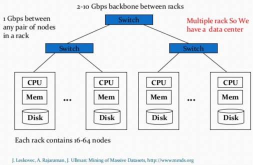
Positionning

- w.r.t. traditional courses on algorithms
  - Exact algorithms for polynomial problems
  - Approximation algorithms for NP-Complete problems
  - Potentially exponential algorithms for difficult problems (going through an ILP for example)
- Here, we will consider extreme contexts
  - not enough space to transmit input data (sketching) or
  - not enough space to store the data stream (streaming)
  - not enough time to use an algorithm other than a linear complexity one
- Compared to the more "classical" context of algorithms:
  - we aim at solving simple problems and
  - we are looking for approximate solutions only because we have very strong time or space constraints.
- Disclaimer: it is not my research topic, but I like to look at the sketching/streaming papers and I am happy to teach it to you!

Application Context 1: Internet of Things (IoT)

- Connected objects, which take measurements
- The goal is to aggregate data.
- Processing can be done either locally, or on their way (fog computing), or in a data center (cloud computing).
- We must be very energy efficient
  - because objects are often embedded without power supply.
- E3nergy cost: Communication is the main source of energy consumption, followed by memory movements (from storage), followed by computations (which are inexpensive)
- A good solution is to do as many local computations as possible!
  - but it is known to be difficult (distributed algorithms)
  - especially when the complexity is not linear (e.g. think about quadratic complexity)
- Solution:
  - compress information locally (and on the fly)
  - only send the summaries; summaries must contain enough information!

Application Context 2: Datacenters



- Aggregate construction
- except the network (we can have several levels + infiniband), everything is "linear"
- the distance between certain nodes/data is very large but a strong proximity with certain data stored on disk
- with 1,000 nodes with 1TB of disk and a link at 400 MB/s, we have 1 PB and 400 GB/s (higher than with a HPC system)
- **provided the data is loaded locally !**
- for 25 TF/s ($10^3$25GFs seti@home) in total, ratio 60 (HPC system 40 000)
- in practice, dedicated to linear algorithms and very inefficient for other classes.
- In both contexts, there is a strong need to have data driven algorithms (where placement is imposed by data) whose complexity is linear

Sketching – Streaming

Sketching - Streaming – Context

- large volume of data generated in a distributed way
  - to be processed locally and compressed before transmission.
- Types of compression?
  - lossless compression
  - compression with losses
  - compression with losses, but controlled tightly controlled loss for a specific function (sketching)
- + we are going to do compression on the fly (streaming)

On-the-fly compression dedicated to a function f

- Easy problems?
  - examples: $\min, \max, \sum$, mean value median?
  - Constraint: linearize the computations (later on plagiarism detection)
- How?
  - The solution is often to switch to **randomized approximation algorithms**.

Compression associated to a specific function f

- More formally, given f,
- we want to compress the data X but still be able to compute $\simeq f(X)$.
- Sketching: we are looking for C_f and g such that
  - the storage space $C_f(X)$ is small (compression)
  - from $f(X)$, we can recover $f(X)$, ie $g(C_f(X)) \simeq f(X)$
- Streaming: additional difficulty, the update is performed on the fly.
  - we cannot compute $C_f(X \bigcup \{y\})$ from $X \bigcup \{y\}$
  - since we cannot store $X \bigcup \{y\}$
  - so we need another function h such that . $h(C_f(X), \{y\}) = C_f(X \bigcup \{y\})$
- and one last difficulty:
- very often, it is impossible to do in deterministic and exact / deterministic and approximate
- but only with a randomized and approximation algorithm.
- How to write this ?
  - We are looking for an estimator Z such that for given α and ϵ
  - $Pr(|Z - f(X)| \geq \epsilon f(X)) \leq \alpha$. How to read this?
    - the probability of making a mistake by a ratio greater than ϵ (as small as you want)
    - is smaller than α (as small as you want)

Example: count the number of visits / packets

- Context
  - a sensor/router sees packets / visits passing through,....
  - you just want to maintain elementary statistics (number of visits, number of visits over the last 1 hour, standard deviations)
  - Here, we simply want to count the number of visits
- What storage is necessary if we have n visits? $\log n$ bits. Why ? Pigeonhole principle. If we have strictly less than $logn$ bits, then we have two events (among the n) that will be coded in the same way.
- What happens if we only allow an approximate answer (say, to a factor of ρ <2)? you need at least $\log \log n$ bits. Why ? sketch of the proof: if we use $t < \log \log n$ bits, then we will be able to distinguish less than $\log n$ different groups and you can estimate how many groups are needed to count $\{0\}, \{0, 1\}, \{0, 1, 2\}, \{0, 1, ..., 7\}$.
- We will look for a randomized and approximated solution
  - Let us set α and ϵ
  - we are looking for an algorithm that computes $\tilde{n}$, an approximation of n
  - that only uses $K \log \log n$ bits storage
  - and such that $Pr(|\tilde{n} - n| \geq \epsilon n) \leq \alpha$
  - K must be a constant...not necessarily a small constant for now!

Crash Course in probabilities

- Z random variable with positive values
- $E(Z)$ is the expectation of Z
- definitions and properties ?
  - $E(Z) = \int \lambda P(Z = \lambda) d\lambda$ or $E(Z) = \sum_j jP(Z = j)$
  - $E(Z) = \int P(Z \geq \lambda) d\lambda$ or $E(Z) = \sum_j P(Z \geq j)$
  - $E(aX + bY) = aE(X) + bE(Y)$
  - total probabilities (with conditioning) $E(Z) = \sum_j E(ZIY = j)P(Y = j)$
- To measure the distance from Z to $E(Z)$, we use the variance $V(Z)$
  - Definition?
  - $V(Z) = E((Z - E(Z))^2) = E(Z^2) - E(Z)^2$
  - Properties:
  - $V(aZ) = a^2 V(Z)$
  - In general, $V(X + Y) \neq V(X) + V(Y)$ (but it is true if X and Y are independent random variables)
- How to measure the difference between Z to $E(Z)$?
  1. Markov: $Pr(Z \geq \lambda) \leq E(Z)/\lambda$
  2. Chebyshev: $Pr(|Z - E(Z)| \geq \lambda E(Z)) \leq \frac{V(Z)}{\lambda^2 E(Z)^2}$
  3. Chernoff: If $Z_1, \ldots, Z_n$ are Independent Bernouilli rv with $p_i \in [0.1]$ and $Z = \sum Z_i$, then
     $Pr(|Z - E(Z)| \geq \lambda E(Z)) \leq 2\exp(\frac{-\lambda^2 E(Z)}{3})$.

Morris Algorithm: Counting the number of events

- Step 1: Find an estimator Z
  - Z must be small (of order of $\log \log n$)
  - we need to define an additional function g
  - such that $E(g(Z)) = n$
- Morris algorithm
  - $Z \to 0$
  - At each event, $Z \to Z + 1$ with probability $1/2^Z$
  - When queried, return $f(Z) = 2^Z - 1$
- What is the space complexity to implement Morris' algorithm?
- What is the time complexity in the worst case? What is the expected complexity of a step?
- Prove the correctness: $E(2^{Z_n} - 1) = n$ (note Z_n the random variable that denotes Z after n events) Hint: by induction, assuming that $E(2^{X_n}) = n + 1$ and showing that $E(2^{X_{n+1}}) = n + 2$
- How to find a probabilistic guarantee of the type $Pr(|f(X_n) = \tilde{n} - n| \geq \epsilon n) \leq \alpha$? Hint Prove $E(2^{2X_n}) = 3/2n^2 + 3/2n + 1$.
- Conclusion? Is this unexpected ?

From Morris to Morris+ and Morris+++

- 2nd step: How to get a useful bound?
- Objective: to reduce the variance (expectation is what we want). How to do it?
    - Classic idea: do the same experience many times and average them
- Morris algorithm +
    - Morris is used to compute independent $Z_n^1, Z_n^2, \ldots, Z_n^K$
    - On demand, compute $Y_n = \sum_i Z_n^i$ return $f(Y_n) = 2^{Y_n} - 1$
- Questions:
    - Which space complexity to implement Morris+'s algorithm?
    - What time complexity?
    - Establish the correctness: $E(2^{X_n} - 1) = n$
    - What is the new guarantee obtained with Chebyshev? How many counters should be maintained?
- How can we do even better?
    - Morris++ = Morris+(1/3) and median
    - proof with Chernoff: If $Z_1, \ldots, Z_n$ are Independent Bernouilli rv with $p_i \in [0.1]$ and $Z = \sum Z_i$, then $Pr(|Z - E(Z)| \geq \lambda E(Z)) \leq 2\exp(\frac{-\lambda^2 E(Z)}{3})$.

2nd example: how to count the number of unique visitors

Context

- It is assumed that visitors are identified by their address ($i_k \in [1, n]$)
- We observe a flow of m visits $i_1, \ldots, i_m$ with $i_k \in [1, n]$
- How many different visitors ?
- Deterministic and trivial algorithms:
  - if n is small, if n is big... and in front of what?
  - solution in n:n bit array
  - solution in $m \log n$: we keep the whole stream!
- We will see a bit later
  - that we cannot do better with exact and deterministic algorithms
  - that we cannot do better with approximated and deterministic algorithms
- How to do if you cannot store n bits
  - but only $O(\log^k n)$ for a certain k?
- we will see that it is again possible by using both randomization and approximation.
- and that no deterministic exact or deterministic approximation can do it with this space constraint.

Idealized algorithm (1) – Flajolet Martin

We will start with an idealized algorithm (which cannot be implemented in practice).

- Let us choose a random h function from $[1, n]$ to $[0, 1]$
- Why idealized?
  - Problem 1: to store such a random function, you must define the images for in each of the n points... at least $\Omega(n)$ bits
  - Problem 2: and in addition we would have to store real values!
  - We will come back to these two problems in a moment....
  - Let us assume for now that storing such a function costs $\Theta(1)$
- How do you keep track of the number of unique visitors?
- We will keep $Z \longrightarrow \min_{i \in \text{stream}} h(i)$. Intuition?
  - If you see the same visitor k times, it won't change Z
  - If we see t different visitors, then the values taken by h split $[0, 1]$ in $t + 1$ intervals...and all should have the same size in expectation... and this size is $\frac{1}{t+1}$ including the first !
- so you should return $\frac{1}{Z} - 1$!

Idealized algorithm (2) – Flajolet Martin

Proof of correctness

- Let's prove that $E(Z) = \frac{1}{t+1}$.
- $E(Z) = \int_0^{+\infty} P(Z \geq \lambda) d\lambda$.
  - Show that $E(Z) = \frac{1}{t+1}$
  - How to continue? by calculating the variance and applying Chebychev
  - Prove that $E(Z^2) = \frac{2}{(t+1)(t+2)}$
  - There is still one foolishness not to be said.... $E(1/Z) \neq 1/E(Z)$
  - Intuition: if we can control closely Z and $\frac{1}{t+1}$, $1/Z - 1$ will be close to t
- FM+
  - Let us maintain $q = \frac{1}{\epsilon^2 \eta}$ FM instances.
  - Z_i is the value produced by FM$_i$
  - What to return? $Y = \frac{1}{(\sum_1^q Z_i)/q} - 1$
  - $E(\frac{\sum_1^q Z_i}{q}) = \frac{1}{t+1}$
  - $V(\frac{\sum_1^q Z_i}{q}) = \frac{t}{q(t+1)^2(t+2)} < \frac{E(Z))^2}{q}$
  - Claim 1: $P(IY - \frac{1}{t+1} I \geq \frac{\epsilon}{t+1}) \leq \eta$
  - Claim 2: $P(I \frac{1}{Y} - 1 - tI \geq \Theta(\epsilon) t) \leq \eta$
- FM++
  - choose $\eta = \frac{1}{3}$adapt ϵ, instantiate K copies of Y $Y_1, \ldots, Y_K$
  - output median$\{\frac{1}{Y_i}\}$ Ok for $K = \lceil 36 \log(\frac{1}{\delta}) \rceil$

Toward a Non Idealized Version. A crucial tool: hashing functions

- We used the set of all possible functions (too large set, to large. storage for one function)
- To make it practical, we will consider a large (not too large) family of functions $\mathcal{H}$ from $[1, p] \rightarrow [1, p]$
- How to define the quality of a family $\mathcal{H}$?
- Notion of k-wise independence
  - $\forall i_1, \ldots, i_k, \forall j_1, \ldots, j_k$, $i_k \neq i_l$, and if we pick a random h function in $\mathcal{H}$, then
  - $P(h(i_1) = j_1$ and $h(i_k) = j_k)$ $1/p^k$
  - a larger k provides a "better" family
- Examples:
  1. the set of all functions from $[1, p] \rightarrow [1, p]$ is Ok.
     - What k, what storage cost?
     - $f(1) \rightarrow p$ choices,..., $f(p) \rightarrow p$ choices
     - Problem: expensive, $p \log p$ bits are necessary for one function
  2. with the polynomials $\mathcal{H}^k_{\text{poly}}$ of degree k in F_p
     - evaluation cost? for degree k, k mult & and adds
     - independence? how many polynomials such that $(h(i_1) = j_1$ and $h(i_k) = j_k$
     - exactly one, Lagrange polynomial: $P = \sum_{r=1}^{k} \frac{\prod_{l \neq r}(X - i_l)}{\prod_{l \neq r}(i_r - i_l)} \times j_r$
     - choice? picking a function at random in $\mathcal{H}^k_{\text{poly}}$ $\rightarrow$ choose $k+1$ coefficients.
     - and thus the family $\mathcal{H}^k_{\text{poly}}$ is $k-$-independent

Non Idealized FM (1)

- Step1: find a $O(1)$-approximation $\tilde{t}$ of t in $O(\log n)$ bits, ie a constant C such that $\frac{t}{C} \leq \tilde{t} \leq Ct$ with constant probability (say $\frac{2}{3}$)
  1. Pick h from a 2-wise family from $[n]$ to $[n]$ (works $\forall n$ but complicated, otherwise round to 2^k, or assume that n is a prime).
  2. Maintain $X = \max_{i \in \text{stream}} lsb(h(i))$ (lsb: least significant bit)
  3. Output 2^X
- Intuition:
  - $P(lsb(h(i)) = j) = \frac{1}{2^{j+1}}$, so $E(\{i, \quad lsb(h(i)) = j\}) = \frac{t}{2^{j+1}}$ and $E(\{i, \quad lsb(h(i)) > j\}) \simeq \frac{t}{2^{j+2}} + \frac{t}{2^{j+3}} + \ldots \simeq \frac{t}{2^{j+1}}$.
  - What happens when j is of order $\log t$...
    - there is $\simeq 1$ visitor such that $lsb(h(i)) = j$
    - there is $\simeq 1$ visitor such that $lsb(h(i)) > j$
  - Thus, if j is of order $(\log t) - 5$ it is very unlikely $(1/2^5)$ that there is no i s.t. $lsb(h(i)) \geq j$
  - Thus, if j is of order $(\log t) + 5$ it is very unlikely $(1/2^5)$ that there is a i s.t. $lsb(h(i)) \geq j$
  - with good probability, $\tilde{t} = 2^X$ is in $[\frac{t}{C}, Ct]$
- The proof is very similar to what we have done, with one tricky issue
  - how to use 2-wise independence ?
  - fix j, define $Y_i = 1$ iff $lsb(h(i)) = j$ so that $Z_j = \sum_i Y_i$, then $E(Z_j) = \frac{1}{2^{j+1}}$
  - as usual we need $V(Z_j)$ to control probabilities and $V(Z_j) = E((\sum_i Y_i)^2) - E(\sum_i Y_i)^2 = \sum V(Y_i) + \sum_{i \neq k} E(Y_i Y_j) - E(Y_i)E(Y_j) = \sum V(Y_i)$ because 2−wise independence says that $E(Y_i Y_j) - E(Y_i)E(Y_j)$!

Non Idealized FM (2)

- Playing with constants, let us assume that Step1 provides a 32-approximation with probability $\frac{2}{3}$, then perform K experiments and take the median to have 32-approx with large probability
- To obtain a stronger approximation, we rely on the following technique
- let us chose g in a 2 wise family from $[n]$ to $[n]$.
  1. Imagine that we consider $\log n$ sets, with $\mathcal{S}_j$ contains the elements i of the stream s.t. $lsb(g(i)) = j$.
  2. we know $\tilde{t}$ (close to t), let us denote by Z the size of $\mathcal{S}_j$ when $2^{j+1} \simeq \tilde{t}\epsilon^2$
  3. and let consider $U = 2^{j+1}Z$ in this case
- $E(U) = 2^{j+1}E(Z) = t$, $V(U_i) = 2^{2j+2}Var(Z) \leq t2^{j+1}$
- so that (Chebychev) $P(IU - tI \geq \epsilon t) \leq \frac{t2^{j+1}}{\epsilon^2 t^2} = \frac{2^{j+1}}{\epsilon^2 \tilde{t}} \frac{\tilde{t}}{t} \leq C'$
- Then, we use several hashing functions and take the average value to obtain an error with arbitrarily small probability
- Not completely finished ! Is this algorithm implementable this time with small space ?
- No, because $\mathcal{S}_0$ is very large for instance ! But the maximum value we are expecting in "interesting" $\mathcal{S}_j$ is $\frac{t}{2^{j+1}} = \frac{\tilde{t}}{2^{j+1}} \frac{t}{\tilde{t}} \leq \frac{C}{\epsilon^2}$
- Thus, we can "only" remember the first $\frac{C}{\epsilon^2}$ is each set !
- Overall space complexity ???

Note on Non Idealized FM (3)

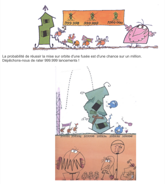
- Technique called Geometric sampling
- n elements in the stream, $k \leq n$ distinct elements (with respect to some property)
- Store $\log n$ sub-streams, where S_0 stores $1/2$ of the elements (distinct wrt the property), S_1 stores $1/4$ of the elements,... $S_{\log k}$ stores (close to) 1 element, $S_{\log n}$ a priori stores nothing if $k << n$
- Suppose that when there are l elements in one of the sets, we can find a good estimation of k where typically l is of order $\frac{1}{\epsilon^2}$
- Then, we bound all the sets to store less than $10l$ elements (they are useless after that)
- if we have a constant approximation of k (obtained elsewhere), then we know in which set we should look at.

Why do we need randomization and approximation?

- Because a deterministic algorithm needs at least $\Omega(n)$ bits
- How to prove this? We assume $n = \Theta(m)$
  - Let us consider the state of the memory of the algorithm after seeing $i_1, \ldots, i_m$
  - We need to prove that there is enough information in what is stored
  - so as to differentiate 2^n distinct elements
  - Remark: you can add as many computations as you want !
  - Input X, let us denote by $C_f(X)$ the state on the memory
  - What can be computed using $C_f(X)$ (and only $C_f(X)$)?
  - we can compute $h(C_f(X))$ and $h(C_f(X), \{y\}) = C_f(X \bigcup \{y\})$
  - do it for all possible y values (visitors)...
  - If y was in the stream, then $h(C_f(X), \{y\}) = h(C_f(X))$ otherwise $h(C_f(X), \{y\}) = h(C_f(X) + 1!$
  - In $C_f(X)$, there is enough information to distinguish 2^n possible vectors (all visitors vectors)
  - and thus n bits are needed!

Why do we need randomization and approximation?

- Because a deterministic approximation algorithm (say 1.1-approx) needs at least $\Omega(n)$ bits
- Let us suppose that there exists a collection $\mathcal{C}$ of subsets of n such that
  - $|\mathcal{C}|$ is large ($\geq \exp(n/10^4)$)
  - $\forall S \in \mathcal{C}, |S| = n/100$ (sets are large)
  - $\forall S_1, S_2 \in \mathcal{C}^2, |S_1 \bigcap S_2| \leq n/2000$ (intersections are small)
- General idea
  - Let us assume that we have presented to the algorithm one of the sequences of $\mathcal{C}$
  - Then, we can find back which one!
  - just by trying exhaustively all $\#\mathcal{C}$ sequences with $C_f(X)$
  - Since we know how to differentiate exponentially many ($\exp(n/10^4)$) elements, we need $\Omega(n)$ bits



- We still need to prove that such a set $\mathcal{C}$ exists !
  - n visitors numbered from 1 to n split into $n/100$ packets of 100 visitors
  - In $S_i, \forall i$ we randomly choose one visitor per packet
  - we build $\exp(n/10^4)$ such sets S_i.
  - easy: What is their size? $n/100$
  - we need to check that $\forall i, j, \ i \neq j, |S_i \bigcap S_j| \leq n/2000$
  - How to do this ?it is enough to prove that the P(it works) is > 0
  - Why does it work ? $Y_{i,j}$ number of collisions between S_i and S_j
  - $E(Y_{i,j})$? $Pr(Y_{i,j} > n/2000)$? $Pr(\exists i, jt.q. Y_{i,j} > n/2000)$?