The role of Riemann–Hilbert techniques in integrable systems, geometry, spectral problems and stochastic point processes

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Abstract A Riemann–Hilbert problem (RHP) is a particular type of boundary value problem for a matrix valued function on the complex plane (or other Riemann surface). It is the analytic tool, for example, to find holomorphic sections of vector bundles, the typical example being the Birkhoff factorization theorem on the Riemann Sphere.
There is a surprisingly wide plethora of problems that can be framed within the theory of RHPs; it includes the inverse spectral problem for integrable wave equations (KdV, mKdV, NLS, AKNS, and, to some extent, KP), the theory of occupation numbers for certain stochastic point fields, the theory of Painlevé equations and even the analysis of the spectral properties of certain inverse problems in tomography.
Special techniques have been developed in the late ?90 to study asymptotic behaviours of solutions of RHPs and this allows rigorous and very (extremely, in fact) detailed asymptotic analysis of nonlinear waves, be it in the long-time or small-dispersion regimes; for example results of "universality" of behaviour of solution near the caustic curve of the zero-dispersion approximation can be approached (if not outright solved) by such techniques.
A "tau" function can be associated to the deformation space of any RHP; in special cases it becomes a Fredholm determinant, in others it takes the meaning of generating function of intersection numbers of characteristic classes on moduli spaces.

In this talk I will try to give an overview of these topics to showcase the breadth and reach of the method, as well as my collaborators' research and my own.

We give an overview of the wide range of applications of Riemann–Hilbert problems in recent results.

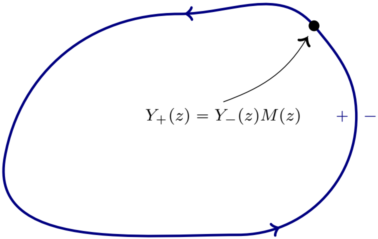
1. Riemann–Hilbert problems
2. Nonlinear Waves (B. Tovbis '15, Grava-Claeys '13, Dubrovin-Grava-Klein '12)
3. Random (Multi)–Matrices (B.-Gekhtman-Szmigielski '12, B. Bothner '15)
4. Spectral properties of tomography (B.-Katsevich-Tovbis, '15–'17).
5. Intersection numbers (B.-Dubrovin-Di '16, B.-Cafasso '16, B.-Ruzza '17)
6. Integrable systems: Painlevé-Calogero-Moser, (Levin-Olshanetsky '00, Takasaki '01, B.-Cafasso-Rubtsov '17)

OPs, NLS, KdV, Gap probatilities, Painlevé equations, etc. are related to a particular type of boundary value problem in the complex plane. A **Riemann–Hilbert problem** is a **boundary–value problem** for a matrix–valued, piecewise analytic function $\Gamma(z)$.

Problem

Let Σ be an **oriented** (union of) curve(s) and $M(z)$ a (sufficiently smooth) matrix function defined on Σ. Find a *matrix-valued* function $Y(z)$ with the properties that

- $Y(z)$ is **analytic** on $\mathbb{C} \setminus \Sigma$;
- $\lim_{z\to\infty} Y(z) = \mathbf{1}$ (or some other normalization);
- $Y_+(z) = Y_-(z)M(z)$; $\forall z \in \Sigma$



In the scalar case, a RHP is reducible to the Sokhotsky-Plemelji formula

Theorem (Sokhotsky-Plemelji formula)

Let $h(w)$ be α–Hölder on Σ and

$$f(z) := \frac{1}{2i\pi}\int_{\Sigma} \frac{h(w)\,\mathrm{d}w}{w-z} \quad \Rightarrow \quad f_+(w) - f_-(w) = h(w) \quad \Rightarrow \quad \mathrm{e}^{f_+} = \mathrm{e}^{f_-}\mathrm{e}^{h} \qquad (1)$$

(Partial) Index problem.

In the matrix case the solution **cannot** be written explicitly (at best an integral equation can be derived) and hence the problem is genuinely transcendental.

Can be rephrased as a triviality of a vector bundle (Birkhoff–theorem ↔ partial indices)

Definition (B. '10, B. '16)

Deformation theory of jump-matrix $M(z;\mathbf{t})$ leads to

$$\partial_t \ln \tau(\mathbf{t}) = \int_\Sigma \mathrm{Tr}\left(Y_-^{-1} Y_-' \partial_t M M^{-1}\right) \frac{\mathrm{d}z}{2i\pi} + \Theta(\mathbf{t}) \qquad (2)$$

Θ is a **smooth** one form of the deformation parameters (an "anomaly"). First term has a **simple pole with integer residue** on the **Malgrange divisor**.

- The principal property: τ is locally analytic and $\tau(\mathbf{t}) = 0$ if and only if the RHP has no solution (vector bundle is non-trivial).
- Behaves like a (regularized) Fredholm determinant (Malgrange '90).
- Connects with symplectic geometry of the deformation space.
- Isomonodromic deformations: Painlevé and conformal blocks in CFT (Iorgov, Lisovyy, Gamayun, Its, Tykhyy,...'13–onwards)

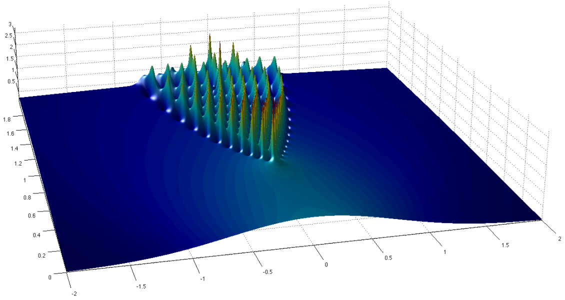
For "oscillatory" problems (depending on parameter), Deift–Zhou method ("non-abelian steepest descent").

Nonlinear Waves

The focusing Nonlinear Schrödinger (NLS) equation,

$$i\hbar\partial_t q \;= -\hbar^2\partial_x^2 q - 2|q|^2 q \tag{3}$$

models self-focusing and self-modulation (*optical fibers*). It is **integrable** by inverse scattering methods (Zakharov–Shabat). It exhibits interesting behaviour as $\hbar \to 0$ (**modulational instability**); in different regions of spacetime, there are different asymptotic behaviors (*phases*) separated by **breaking curves** (or **nonlinear caustics**).



The tip-point of the braking curves is called a point of **gradient catastrophe**, or **elliptic umbilical singularity** [Dubrovin-Grava-Klein].

Main goal

Leading order asymptotic $q(x,t,\hbar)$ on and around the gradient catastrophe point (x_0,t_0).

The behavior in the bulk is described in terms of slow modulation of exact quasi-periodic solutions (**genus 2**), while outside by slow modulation equations for the amplitude. There are (generically) two types of **transitional regions**

- A strip region of scale $\mathcal{O}(\hbar\ln\hbar)$ around the *breaking curves* (nonlinear caustics);
- a circular region of scale $\mathcal{O}(\hbar^{\frac{4}{5}})$ around the gradient catastrophe point.

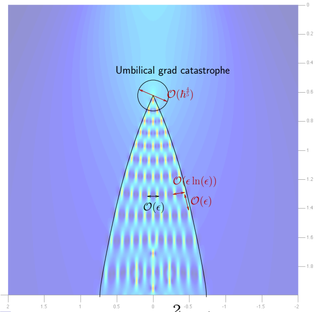


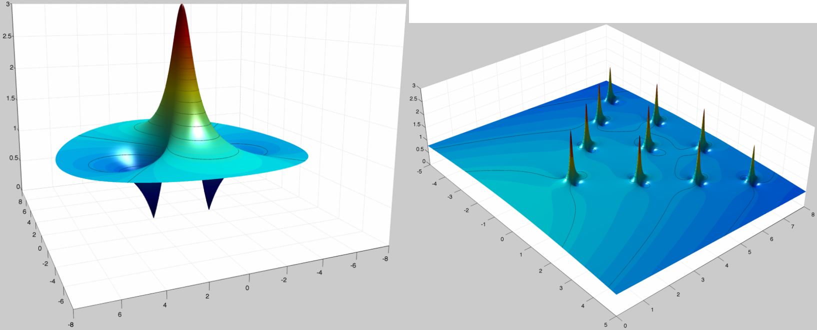
FIGURE: $A(x)=e^{-x^2}$, $\Phi'(x)=\tanh x$ and $\hbar=0.03$

If we scale by $\hbar$ around each peak we find the *rational or Peregrine breather*

$$\xi = \frac{x - x_p}{\hbar} \;, \quad \eta = \frac{x - x_p}{\hbar} \tag{4}$$

$$Q_{br}(\xi,\eta) = \mathrm{e}^{-2i\left(a\xi + (2a^2 - b^2)\eta\right)} b \left(1 - 4\frac{1 + 4ib^2\eta}{1 + 4b^2(\xi + 4a\eta)^2 + 16b^4\eta^2}\right) \tag{5}$$

$$i\partial_\eta Q_{br} + \partial_\xi^2 Q_{br} + 2|Q_{br}|^2 Q_{br} = 0 \tag{6}$$



Emergence of "Peregrine breather solution" and time of highest peak after shock predicted analytically in [B.-Tovbis '13 CPAM] and experimentally verified in nonlinear optics:
"Universality of the Peregrine Soliton in the Focusing Dynamics of the Cubic Nonlinear Schrödinger Equation", Phys. Rev. Lett. **119** (2017)
Tikan-Billet-El-Tovbis-B.-Sylvestre-Gustave-Randoux-Genty-Suret-Dudley

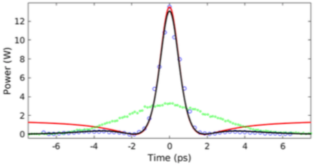


FIG. 4. Experimental and numerical simulations: temporal dynamics of the optical power (setup 1). Input pulse (green points) corresponding to $N = 1/\epsilon \simeq 2.2$. Output of the 400 m-long PMF (blue circles). Numerical simulations of the NLSE (black line) and theoretical Peregrine soliton (red line).

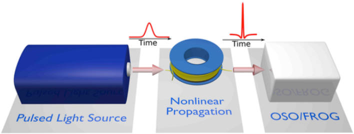


FIG. 3. Schematic experimental setups. The pulsed light source is either a fiber picosecond laser or a spectrally filtered femtosecond OPO. The nonlinear propagation of pulses is achieved in a HNLF or in a standard PMF fiber.

The KdV equation

$$u_t = uu_x + \epsilon^2 u_{xxx}\ , \qquad u(x,0) = u_0(x) \qquad \text{rapidly decaying} \tag{7}$$

For $\epsilon = 0$ we have Burger's equation $u_t = uu_x$, solved by the hodograph method (characteristics), locally

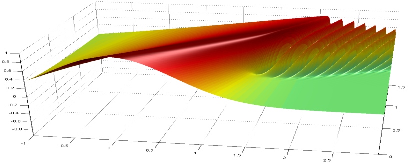
$$f(u) = x + u\,t \qquad f(u) = u_0^{-1} \tag{8}$$

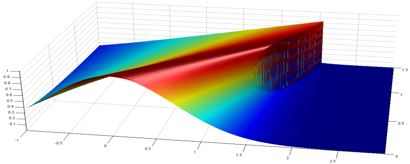
It shocks at $t_0 = \frac{1}{\max u_0'(x)}$.
The small-dispersion also exhibits interesting behavior:

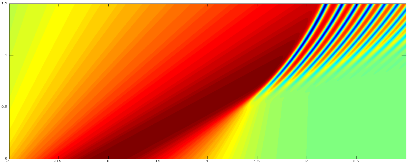
- Near the point of gradient catastrophe (x_0, t_0) its behavior is described in terms of a generalization of the Painlevé I equation with critical scale $\hbar^{\frac{6}{7}}$ (Dubrovin-Grava-Klein, Grava-Claeys);
- Near the trailing edge (after the time t_0) it is described by the Hastings-McLeod solution of the Painlevé II equation $y''(s) = sy(s) + 2y^3(s)$ with critical scale $\hbar^{\frac{2}{3}}$ (Grava-Claeys);
- Near the leading edge the behavior is described in terms of elementary function (superposition of soliton solutions) with scale $\hbar \ln \hbar$ (Grava-Claeys)

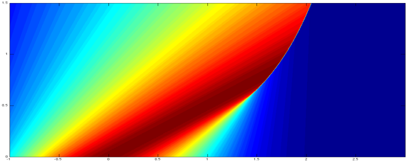
KdV-small dispersion

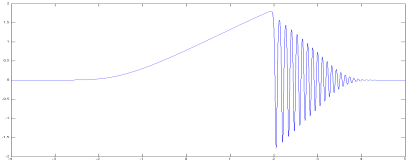
KdV-zero dispersion = Burgers

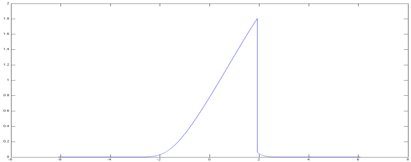


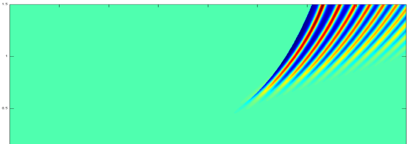


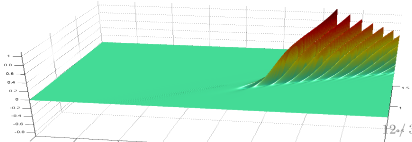












The nonlinear Schrödinger equation (in 1 spatial dimension)

$$i\hbar q_t(x,t) = -\hbar^2 q_{xx}(x,t) \pm 2|q(x,t)|^2 q(x,t) \tag{9}$$

Theorem (Zakharov)

Let $\Gamma(z;x,t)$ be a 2×2 matrix, analytic in $z \in \mathbb{C}\backslash\mathbb{R}$,

$$\Gamma_+(z;x,t) = \Gamma_-(z;x,t)\begin{bmatrix} 1-|r(z)|^2 & -\overline{r}(z)\mathrm{e}^{-\frac{2i}{\hbar}(2tz^2+xz)} \\ r(z)\mathrm{e}^{\frac{2i}{\hbar}(2tz^2+xz)} & 1 \end{bmatrix} \tag{10}$$

$$\Gamma(z;x,t) = \mathbf{1} + \mathcal{O}(z^{-1})\ , \quad |z|\to\infty \tag{11}$$

Then *the function of x,t*

$$q(x,t) := 2i\lim_{z\to\infty} z\Gamma_{12}(z;x,t) \tag{12}$$

is a solution of the defocusing NLS, with initial data given by the data that was associated to the scattering transform.

The **advantage** of the formulation of the Theorem is that the x,t dependence is in plain sight; the **disadvantage** is that it is not possible (in general) to obtain a closed formula for the solution of the advocated Riemann–Hilbert problem.

Random Matrices

The typical setup: $\mathcal{H}_N := \{M$ Hermitean $N \times N$ matrix $(M = M^\dagger)\}$.

$$d\mu := dM e^{-\mathrm{tr} V(M)} \tag{13}$$

$$dM = \prod_{i<j} d\Re(M_{ij})\, d\Im(M_{ij}) \prod_{k} dM_{kk} \tag{14}$$

$$Z_N^{1MM}[V] := \int d\mu = \text{ Partition function.} \tag{15}$$

B.-EYNARD-HARNAD ('06)

$$Z_N^{1MM}[V] = \tau[V] \,, \qquad V(z) = \sum_j t_j z^j$$

DEFINITION (THE p-CHAIN-CAUCHY MATRIX-MODEL, B.-GEKHTMAN-SZMIGIELSKI '10-13, B.-BOTHNER '15)

Let $\mathcal{M}_{n,+}^{(p)}$ be the set of p–tuples of *positive semidefinite* Hermitean matrices with the following class of measures

$$\mathrm{d}\mu(\vec{M}) = \mathcal{Z}\frac{\prod_{j=1}^{p} \det(M_j)^{a_j} \mathrm{e}^{-N \operatorname{Tr} U_j(M_j)} \, \mathrm{d}M_j}{\prod_{j=1}^{p-1} \det(M_j + M_{j+1})^n} \tag{16}$$

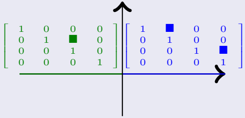
The scaling parameter N is taken proportional to the size when considering the limit of large sizes $n \to \infty$.

No explicit formulas for finite size n, $p \geqslant 3$; however

THEOREM (RIEMANN-HILBERT CHARACTERIZATION FOR $\{\psi_k, \phi_k\}_{k\geqslant 0}$)

Determine a $(p+1)\times(p+1)$ *function* $\Gamma(z) = \Gamma_n(z)$ *with jump on* $\mathbb{R}$ ($\mu_j(z) = z^{a_j}\mathrm{e}^{-U_j(z)}$)*;*

$$\Gamma_+(z) = \Gamma_-(z)\begin{pmatrix} 1 & \mu_1(z)\chi_+ & 0 & & \\ 0 & 1 & \mu_2(-z)\chi_- & 0 & \\ & & 1 & \dots & \\ & & & & 1 \end{pmatrix}$$



$$\Gamma(z) = \left(\mathbf{1} + \mathcal{O}\left(z^{-1}\right)\right)\mathrm{diag}\left[z^n, 1, \dots, 1, z^{-n}\right], \quad z \to \infty.$$

Moreover the correlation kernels $\mathbb{K}_{j\ell}$ *are given by (B.-Bothner '15)*

$$\mathbb{K}_{j\ell}(x,y) = e^{-\frac{1}{2}U_j(x) - \frac{1}{2}U_\ell(y)}\mathbb{M}_{j\ell}(x,y),$$

$$\mathbb{M}_{j\ell}(x,y) = \frac{(-)^{\ell-1}}{(-2\pi i)^{j-\ell+1}}\left[\frac{\Gamma^{-1}(w;n)\Gamma(z;n)}{w-z}\right]_{j+1,\ell}\Bigg|_{\substack{w=x(-)^{j+1}\\ z=y(-)^{\ell-1}}}$$

Conjecture (B.Bothner '15)

For any $p = 2, 3, \ldots$, there exists $c_0 = c_0(p)$ and $\{\eta_j\}_1^p$ which depend on $\{a_j\}_1^p$ such that

$$\lim_{n\to\infty} \frac{c_0}{n^{p+1}} n^{\eta_\ell - \eta_j} \mathbb{K}_{j\ell}\left(\frac{c_0}{n^{p+1}}\xi, \frac{c_0}{n^{p+1}}\eta\right) \propto \mathcal{G}_{j\ell}^{(p)}\left(\xi, \eta; \{a_j\}_1^p\right)$$

uniformly for ξ, η chosen from compact subsets of $(0, \infty)$. Here the limiting correlation kernels equal

$$\mathcal{G}_{j\ell}^{(p)}\left(\xi, \eta; \{a_j\}_1^p\right) = \int_L \int_{\hat{L}} \frac{\prod_{s=0}^{\ell-1} \Gamma(u - a_{1s})}{\prod_{s=\ell}^{p} \Gamma(1 + a_{1s} - u)} \frac{\prod_{s=j}^{p} \Gamma(a_{1s} - v)}{\prod_{s=0}^{j-1} \Gamma(1 - a_{1s} + v)} \frac{\xi^v \eta^{-u}}{1 - u + v} \frac{dv\, du}{(2\pi i)^2}$$
$$+ \sum_{s\in\mathcal{P}\cup\{0\}} \operatorname*{res}_{v=s} \frac{\prod_{s=0}^{\ell-1} \Gamma(1 + v - a_{1s})}{\prod_{s=\ell}^{p} \Gamma(a_{1s} - v)} \frac{\prod_{s=j}^{p} \Gamma(a_{1s} - v)}{\prod_{s=0}^{j-1} \Gamma(1 + v - a_{1s})} \frac{\xi^v \eta^{-v}}{(-)^j \xi - (-)^\ell \eta}$$

with $\mathcal{P} = \{a_{1\ell} := \sum_{j=1}^{\ell} a_j,\ 1 \leqslant \ell \leqslant p\}$.

Remark

Found also in the statistical analysis of singular values of products of Ginibre random matrices (Akemann–Burda '12, A.Kieburg-Wei '13, Kuijlaars–Zhang '13) (the $(1, 1)$ entry specifically) of the kernels.

Random Matrices and Intersection Numbers

The *Kontsevich–Penner matrix integral* is

$$\mathcal{Z}_n(Y;N) := \frac{\det(\imath Y)^N}{\int_{H_n} \mathrm{d}M \exp \mathrm{Tr}\left(-YM^2\right)} \int_{H_n} \mathrm{d}M \, \frac{\exp \mathrm{Tr}\left(\frac{1}{3}M^3 - YM^2\right)}{\det(M+\imath Y)^N}. \tag{17}$$

$$\log \mathcal{Z}_n(Y;N) \sim \log \mathcal{Z}(\mathbf{t};N), \quad T_k = \begin{cases} \frac{1}{k}\mathrm{Tr}\left(Y^{-k}\right) & k = 1, \ldots, n \\ 0 & k \geqslant n+1 \end{cases} \tag{18}$$

$$t_k := (-1)^k \, k!! \, 2^{-\frac{k}{3}} \, T_k \tag{19}$$

Conjecture (Alexandrov-Buryak-Tessler '17, Pandharipande-Solomon-Tessler '1)

The coefficients of the formal power series $\log \mathcal{Z}(\mathbf{t};N)$ are the *open intersection numbers*.

The open intersection numbers are a generalization of the closed intersection numbers:

$$\langle \tau_{r_1} \cdots \tau_{r_n} \rangle_c := \int_{\overline{\mathcal{M}_{h,n}}} \psi_1^{r_1} \wedge \cdots \wedge \psi_n^{r_n} \tag{20}$$

$$\left\langle \tau_{\frac{d_1}{2}} \cdots \tau_{\frac{d_n}{2}} \right\rangle_o = \frac{\partial^n}{\partial t_{d_1+1} \cdots \partial t_{d_n+1}} \log \mathcal{Z}(\mathbf{t}; N) \Big|_{\mathbf{t}=0} \tag{21}$$

which would be a generalization of the Kontsevich's identity;

$$\begin{gathered}\sum_{m\geqslant 0} \frac{\Gamma\left(\frac{a-b+1}{2}\right)}{\Gamma\left(\frac{a-b+1+6m}{2}\right)} P_{a,b}^{2m}(N) Z^m = \mathrm{e}^{\frac{Z}{3}} \, {}_2F_2\left(\begin{matrix} \frac{1-a-b-2N}{2} & \frac{1+a+b+2N}{2} \\ \frac{1}{2} & \frac{1+a-b}{2} \end{matrix} \middle| -\frac{Z}{4} \right) \\ \sum_{m\geqslant 0} \frac{\Gamma\left(\frac{a-b+2}{2}\right)}{\Gamma\left(\frac{a-b+4+6m}{2}\right)} P_{a,b}^{2m+1}(N) Z^m = -\frac{2N+a+b}{2} \mathrm{e}^{\frac{Z}{3}} \, {}_2F_2\left(\begin{matrix} \frac{2-a-b-2N}{2} & \frac{2+a+b+2N}{2} \\ \frac{3}{2} & \frac{2+a-b}{2} \end{matrix} \middle| -\frac{Z}{4} \right)\end{gathered} \tag{22}$$

$$A(\lambda) := \begin{bmatrix} N \sum_{k\geqslant 0} P_{1,-1}^k(N) \lambda^{-\frac{3k+2}{2}} & \sum_{k\geqslant 0} P_{-1,-1}^k(N) \lambda^{-\frac{3k}{2}} & \sum_{k\geqslant 0} P_{0,-1}^k(N) \lambda^{-\frac{3k+1}{2}} \\ N \sum_{k\geqslant 0} P_{1,0}^k(N) \lambda^{-\frac{3k+1}{2}} & \sum_{k\geqslant 0} P_{-1,0}^k(N) \lambda^{-\frac{3k-1}{2}} & \sum_{k\geqslant 0} P_{0,0}^k(N) \lambda^{-\frac{3k}{2}} \\ N \sum_{k\geqslant 0} P_{1,1}^k(N) \lambda^{-\frac{3k}{2}} & \sum_{k\geqslant 0} P_{-1,1}^k(N) \lambda^{-\frac{3k-2}{2}} & \sum_{k\geqslant 0} P_{0,1}^k(N) \lambda^{-\frac{3k-1}{2}} \end{bmatrix}. \tag{23}$$

EXAMPLE: ONE–POINT OPEN INTERSECTION NUMBERS

$$\sum_{d_1,\dots,d_n\geqslant 0}\left\langle \prod_{i=1}^{n}\tau_{\frac{d_i}{2}}\frac{(-1)^{d_i+1}(d_i+1)!!}{2^{\frac{d_i+1}{3}}\lambda_i^{\frac{d_i}{2}+1}}\right\rangle_o = \tag{24}$$

$$= \begin{cases} -\sum_{g\geqslant 1}\frac{2}{3g+2}P_{0,0}^{g+1}(N)\lambda_1^{-\frac{3g+1}{2}} & n=1 \\ -\frac{1}{n}\sum_{i\in S_n}\mathrm{Tr}\,\frac{A(\lambda_{i_1})\cdots A(\lambda_{i_n})}{(\lambda_{i_1}-\lambda_{i_2})\cdots(\lambda_{i_n}-\lambda_{i_1})}-\frac{\delta_{n,2}}{\left(\lambda_1^{\frac{1}{2}}-\lambda_2^{\frac{1}{2}}\right)^2} & n\geqslant 2. \end{cases} \tag{25}$$

GENERATING FUNCTION OF ONE–POINT I.N. [B.-RUZZA '17]

Itzykson–Zuber '92

$$\sum_{r\geqslant 0}\langle\tau_{r-2}\rangle_c X^r = \mathrm{e}^{\frac{X^3}{24}} \quad\Rightarrow\quad \langle\tau_{3h-2}\rangle_c = \frac{1}{24^h h!}$$

B.–Ruzza '17

$$\sum_{d\geqslant 0}\langle\tau_{\frac{d}{2}-2}\rangle_o X^{\frac{d}{2}} = \mathrm{e}^{\frac{X^3}{6}}\left({}_2F_2\left(\begin{matrix}\frac{1}{2}-N & \frac{1}{2}+N\\ \frac{1}{2} & \frac{1}{2}\end{matrix}\middle| -\frac{X^3}{8}\right)\right.$$
$$\left.+N X^{\frac{3}{2}}\,{}_2F_2\left(\begin{matrix}1-N & 1+N\\ 1 & \frac{3}{2}\end{matrix}\middle| -\frac{X^3}{8}\right)\right)$$

(Noncommutative) Painlevé equations

Paul Painlevé studied (1900) and classified all **second order** ODEs

$$x'' = R(x', x, t) \tag{26}$$

with R a rational function, such that the only moveable singularities of the solutions are poles (i.e. not essential singularities or branchpoint).
50 canonical forms; 6 genuinely transcendental (not reducible to known ODEs and special functions).

P-I

$$x'' = 6x^2 + t \tag{27}$$

P-II

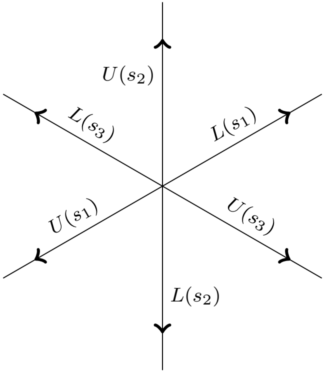
$$x'' = 2x^3 + xt + \alpha \tag{28}$$

P-III

$$t\,x\,x'' = t(x')^2 - xx' + \delta t + \beta x + \alpha x^3 + \gamma t x^4 \tag{29}$$

Etc.

All the Painlevé equations are related to a Riemann–Hilbert problem. For example P-II



$$L(s) := \begin{bmatrix} 1 & 0 \\ s\,e^{\frac{i4}{3}z^3 + ixz} & 1 \end{bmatrix},$$

$$U(s) := \begin{bmatrix} 1 & s\,e^{-\frac{i4}{3}z^3 - ixz} \\ 0 & 1 \end{bmatrix}$$

$$s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0$$

$$\Gamma(z) \sim \mathbf{1} + \mathcal{O}(z^{-1})$$

$$u = u(x;\vec{s}) = 2 \lim_{z\to\infty} z\,\Gamma_{12}(z;x,\vec{s})$$

$$t(t-1)H_{VI} = \mathrm{Tr}\Big[\mathbf{q}(\mathbf{q}-1)(\mathbf{q}-t)\mathbf{p}^2 + \\ +\Big((\theta^0+1-[\mathbf{p},\mathbf{q}])\mathbf{q}(\mathbf{q}-1)+\theta^t(\mathbf{q}-1)(\mathbf{q}-t)+(\theta+2\theta_1^\infty-1)\mathbf{q}(\mathbf{q}-t)\Big)\mathbf{p}+ \\ +(\theta+\theta_1^\infty)(\theta^0+\theta^t+\theta_1^\infty)\mathbf{q}\Big]$$

$$tH_V = \mathrm{Tr}\Big[\mathbf{p}(\mathbf{p}+t)\mathbf{q}(\mathbf{q}-1)+\beta\mathbf{p}\mathbf{q}+\gamma\mathbf{p}-(\alpha+\gamma)t\mathbf{q}\Big],$$

$$tH_{IV} = \mathrm{Tr}\Big[\mathbf{p}\mathbf{q}(\mathbf{p}-\mathbf{q}-t)+\beta\mathbf{p}+\alpha\mathbf{q}\Big],$$

$$tH_{III(D6)} = \mathrm{Tr}\Big[\mathbf{p}^2\mathbf{q}^2-(\mathbf{q}^2-\beta\mathbf{q}-t)\mathbf{p}-\alpha\mathbf{q}\Big],$$

$$tH_{III(D7)} = \mathrm{Tr}\Big[\mathbf{p}^2\mathbf{q}^2+\alpha\mathbf{p}\mathbf{q}+t\mathbf{p}+\mathbf{q}\Big],$$

$$tH_{III(D8)} = \mathrm{Tr}\Big[\mathbf{p}^2\mathbf{q}^2+\mathbf{p}\mathbf{q}-\mathbf{q}-t\mathbf{q}^{-1}\Big],$$

$$tH_{II} = \mathrm{Tr}\Big[\mathbf{p}^2-(\mathbf{q}^2+t)\mathbf{p}-\alpha\mathbf{q}\Big],$$

$$tH_I = \mathrm{Tr}\Big[\mathbf{p}^2-\mathbf{q}^3-t\mathbf{q}\Big].$$

$$\Omega = \mathrm{Tr}\,(d\mathbf{p}\wedge d\mathbf{q}) \qquad [\mathbf{p},\mathbf{q}] = \mathrm{const}$$

General procedure:

- We start with a Lax pair of type

$$\begin{cases} \frac{\partial}{\partial z}\Psi(z;t) = A(z;\mathbf{q},\mathbf{q}^{-1},\mathbf{p},t)\Psi(z;t) \\ \frac{\partial}{\partial t}\Psi(z;t) = B(z;\mathbf{q},\mathbf{q}^{-1},\mathbf{p},t)\Psi(z;t), \end{cases} \Longrightarrow \dot{\mathbf{q}} = \mathcal{A}(\mathbf{q},\mathbf{p},t), \qquad \dot{\mathbf{p}} = \mathcal{B}(\mathbf{q},\mathbf{p},t)$$

  with $\mathcal{A}, \mathcal{B}$ polynomials (rationals) in $\mathbf{q}, \mathbf{p}$ such that the equations above are Hamiltonians and

  $\mathcal{A}$ is of degree at most 1 in $\mathbf{p}$ and $\mathcal{B}$ is of degree at most 2 in $\mathbf{p}$.

- $[\mathbf{p}, \mathbf{q}]$ is a constant of motion: flow preserves coadjoint orbits
- On special Kazhdan-Kostant-Sternberg orbit

$$[\mathbf{p},\mathbf{q}] = ig\begin{bmatrix} 0 & 1 & \dots 1 \\ 1 & 0 & 1\dots \\ 1 & \dots 1 & 0 \end{bmatrix} \tag{30}$$

$$\Longrightarrow \begin{cases} \dot{X} & = & \mathcal{A}(X,Y,t) + [X,F], \\ \dot{Y} & = & \mathcal{B}(X,Y,t) + [Y,F]. \end{cases}$$

$$X = \mathrm{Diag}(q_1,\dots,q_n), \qquad Y = \mathrm{Diag}(p_1,\dots,p_n) + \left(\frac{ig}{q_j - q_k}\right)_{j\neq k}$$

,

Proposition :

$$(x_i - x_j)^2 F_{i,j} = \Big([\mathcal{A}(X,Y),X]\Big)_{i,j}, \quad i \neq j,$$

$$F_{jj} = -\sum_{k:k\neq j} F_{jk} + K, \quad K := \frac{1}{n} \sum_{\ell,m:\ell\neq m} F_{\ell,m}.$$

All entries of F are rational functions of $(x_1,\dots,x_n)$ only.

Proof :

$$[X,\dot{X}] = 0 \Longrightarrow \Big[X,[X,F]\Big] = \Big[\mathcal{A}(X,Y),X\Big]. \quad \text{(This gives the first equation).}$$

$$0 = \frac{d}{dt}[X,Y] = [\mathcal{A}(X,Y),X] + [Y,\mathcal{B}(X,Y)] + \Big(\big[[X,F],Y\big] + \big[X,[Y,F]\big]\Big).$$

On the other hand

$$[\mathcal{A}(\mathbf{q},\mathbf{p}),\mathbf{p}] + [\mathbf{p},\mathcal{B}(\mathbf{q},\mathbf{p})] = 0 \Longrightarrow [\mathcal{A}(X,Y),X] + [Y,\mathcal{B}(X,Y)] = 0.$$

Hence

$$0 = \big[[X,F],Y\big] + \big[X,[Y,F]\big] = -\big[[Y,X],F\big] = [ig(v^T v),F]$$

The off–diagonal entries of the equation above give the linear system of equations

$$f_i + \sum_{j\neq i} F_{i,j} - f_k - \sum_{j\neq k} F_{j,k} = 0, \quad i,k = 1,\dots,n;\ i \neq k.$$

From non-commutative Painlevé to Calogero–Painlevé systems.

(Using Takasaki ('10) canonical transformations.)

$$H_{VI}: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}+\sum_{n=0}^{3} g_n^2\wp(q_j+\omega_n)\right)+g^2\sum_{j\neq k}\Big(\wp(q_j-q_k)+\wp(q_j+q_k)\Big).$$

$$H_V: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}-\frac{\alpha}{\sinh^2(q_j/2)}-\frac{\beta}{\cosh^2(q_j/2)}+\frac{\gamma t}{2}\cosh(q_j)+\frac{\delta t^2}{8}\cosh(2q_j)\right)+$$
$$+g^2\sum_{j\neq k}\left(\frac{1}{\sinh^2((q_j-q_k)/2)}+\frac{1}{\sinh^2((q_j+q_k)/2)}\right).$$

$$H_{IV}: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}-\frac{1}{2}\left(\frac{q_j}{2}\right)^6-2t\left(\frac{q_j}{2}\right)^4-2(t^2-\alpha)\left(\frac{q_j}{2}\right)^2+\beta\left(\frac{q_j}{2}\right)^{-2}\right)+$$
$$+g^2\sum_{j\neq k}\left(\frac{1}{(q_j-q_k)^2}+\frac{1}{(q_j+q_k)^2}\right).$$

$$H_{III}: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}-\frac{\alpha}{4}e^{q_j}+\frac{\beta t}{4}e^{-q_j}-\frac{\gamma}{8}e^{2q_j}+\frac{\delta t^2}{8}e^{-2q_j}\right)+g^2\sum_{j\neq k}\frac{1}{\sinh^2\left((q_j-q_k)/2\right)}.$$

$$H_{II}: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}-\frac{1}{2}\left(q_j^2+\frac{t}{2}\right)^2-\alpha q_j\right)+g^2\sum_{j\neq k}\frac{1}{(q_j-q_k)^2}.$$

$$H_I: \quad \sum_{j=1}^{\ell}\left(\frac{p_j^2}{2}-2q_j^3-tq_j\right)+g^2\sum_{j\neq k}\frac{1}{(q_j-q_k)^2}.$$

$$\frac{d}{dz}\Psi(\mathbf{t};z) = A(\mathbf{t};z)\Psi(\mathbf{t};z); \quad A(\mathbf{t};z) := \begin{pmatrix} i\frac{z^2}{2} + i\mathbf{q}^2 + i\frac{\mathbf{t}}{2} & z\mathbf{q} - i\mathbf{p} - \frac{\theta}{z} \\ z\mathbf{q} + i\mathbf{p} - \frac{\theta}{z} & -i\frac{z^2}{2} - i\mathbf{q}^2 - i\frac{\mathbf{t}}{2} \end{pmatrix}.$$

There exists a unique piecewise analytic solution $\Psi = \{\Psi_\nu,\ \nu = 0,\dots,7\}$ satisfying

$$\Psi(\mathbf{t};z) \sim \left(\mathbf{1} + \frac{\alpha_1 \otimes \sigma_3 - \mathbf{q}\otimes\sigma_2}{z} + \mathcal{O}(z^{-2})\right) e^{(\ln z + i\pi\epsilon)[\mathbf{q},\mathbf{p}]\otimes\mathbf{1}} e^{\frac{i}{2}\left(\frac{z^3}{3} + \mathbf{t}z\right)\hat{\sigma}_3},$$

The corresponding (matrix) Stokes operator $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ satisfy the **noncommutative** relations

$$(\mathbf{X} + \mathbf{Z} + \mathbf{XYZ})Q + Q^{-1}\mathbf{Y} = 2i\sin(\pi\theta)$$

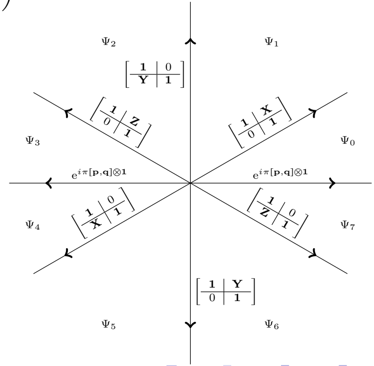
$$(\mathbf{XY} + \mathbf{1})Q - Q^{-1}(\mathbf{YX} + \mathbf{1}) = 0$$

$$\mathbf{Z}Q\mathbf{X} - \mathbf{X}Q^{-1}\mathbf{Z} + Q - Q^{-1} = 0$$

$$(\mathbf{YZ} + \mathbf{1})Q - Q^{-1}(\mathbf{ZY} + \mathbf{1}) = 0$$

$$\mathbf{Y}Q + Q^{-1}(\mathbf{X} + \mathbf{Z} + \mathbf{ZYX}) = 2i\sin(\pi\theta),$$

$$Q := e^{i\pi[\mathbf{p},\mathbf{q}]}.$$



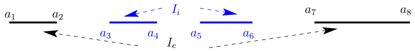
If $Q = \mathrm{e}^{i\pi[\mathbf{p},\mathbf{q}]} = \pm 1$ then we recover the classical case:

$$[\mathbf{X},\mathbf{Y}] = [\mathbf{X},\mathbf{Z}] = [\mathbf{Y},\mathbf{Z}] = 0$$
$$\mathbf{X}+\mathbf{Y}+\mathbf{Z}+\mathbf{XYZ} = \mathrm{const},$$

For $[\mathbf{p},\mathbf{q}] = i\hbar$ (only operatorial setting) we obtain "quantized" Stokes manifold, giving explicit presentation to (Mazzocco-Rubtsov '12).

Spectral asymptotics in Tomography and inverse problems

In tomography applications with partial data:



Theorem

The integral operator $(\hat{K}\phi)(z) = \int_I K(z,x)\phi(x)dx$ *from* $L^2(I)$ *to* $L^2(I)$*, where*

$$K(z,x) = \frac{w^{\frac{1}{2}}(x)w^{-\frac{1}{2}}(z)\chi_e(z)\chi_i(x) + w^{\frac{1}{2}}(z)w^{-\frac{1}{2}}(x)\chi_i(z)\chi_e(x)}{2i\pi(x-z)}, \tag{31}$$

or equivalently

$$\widehat{K}\big|_{I_i} f = \int_{I_i} \sqrt{\frac{w(z)}{w(x)}} \frac{f(x)\,\mathrm{d}x}{2i\pi(x-z)} \chi_{I_e}(x)\ , \quad \widehat{K}\big|_{I_e} g = \int_{I_e} \sqrt{\frac{w(x)}{w(z)}} \frac{g(z)\,\mathrm{d}z}{2i\pi(x-z)} \chi_{I_i}(z) \tag{32}$$

where

$$w(x) := \sqrt{|x-a_1||x-a_{2g+2}|} \tag{33}$$

The function $w(x)$ *can be replaced by almost any smooth positive function.*

Theorem

The operator $\widehat{K}$ is self-adjoint and Hilbert-Schmidt (in fact even Trace-Class); the spectrum is $\{\pm\lambda_n\}$ with $\lambda_1 > \lambda_2 > \dots$. The eigenvalues are simple. In fact $\widehat{K}\widehat{K}^\dagger$ is the direct sum of two endomorphisms of $L^2(I_i), L^2(I_e)$ both of which are **totally positive**.

Because of the underlying total positivity, the eigenfunctions satisfy a sort of Sturm-theorem, by which they change signs as many times as the ordinal of the eigenvalue.

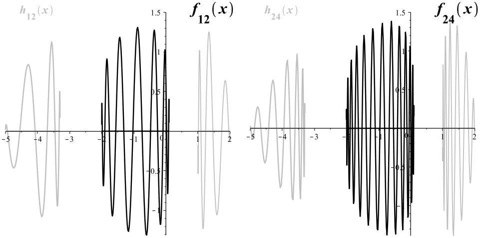


Figure: Two pairs of the corresponding singular functions (f_{12}, h_{12}) and (f_{24}, h_{24}), obtained numerically simultaneously with λ_n. Note, the envelope of the oscillations is already visibly the same, as expected from the asymptotic description below.

Problem

Find a 2×2 matrix-function $\Gamma = \Gamma(z;\lambda)$, $\lambda \in \mathbb{C}\backslash\{0\}$, which is analytic in $\overline{\mathbb{C}}\backslash I$, where $I = I_i \cup I_e$, admits non-tangential boundary values from the upper/lower half-planes that belong to L^2_{loc} in the interior points of I, and satisfies

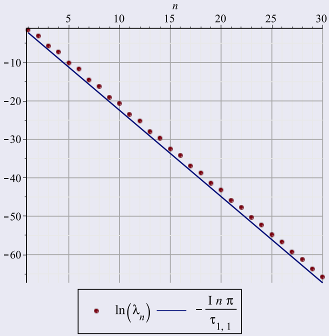
$$\Gamma_+(z;\lambda) = \Gamma_-(z;\lambda)\begin{bmatrix} 1 & 0 \\ \frac{iw}{\lambda} & 1 \end{bmatrix}, \quad z \in I_i; \qquad \Gamma_+(z;\lambda) = \Gamma_-(z;\lambda)\begin{bmatrix} 1 & -\frac{i}{\lambda w} \\ 0 & 1 \end{bmatrix}, \quad z \in I_e, \tag{34}$$

$$\Gamma(z;\lambda) = \mathbf{1} + O(z^{-1}) \quad \text{as} \quad z \to \infty, \tag{35}$$

Theorem (B.-Katsevich-Tovbis '14)

The spectrum of the operator corresponds to values of λ for which the RHP above has no solution; asymptotically

$$\lambda_n = e^{-n\frac{i\pi}{\tau_{11}} + \mathcal{O}(1)}. \qquad (36)$$



Theorem (B.-Katsevich-Tovbis '17, in progress)

If ℓ of the gaps shrink to zero, the spectrum becomes continuous in $[0, 1]$ with multiplicity 2ℓ.