Random tilings and random matrices

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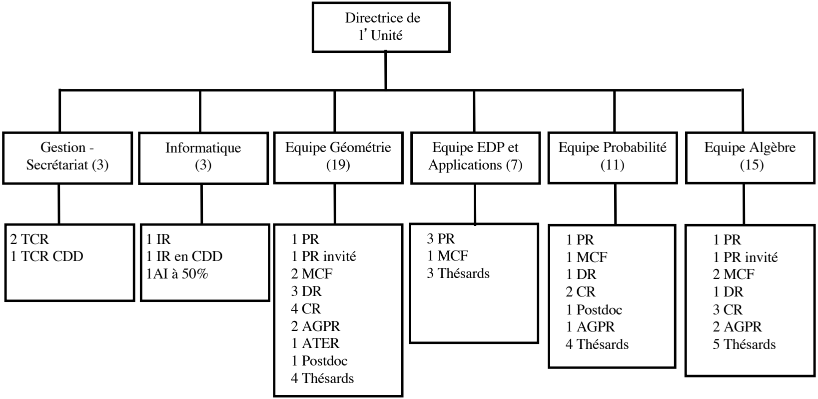

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Probability Team

Grégory Miermont

Random planar maps

Emmanuel Jacob

Random graphs and Processes

Adrien Kassel

Combinatorial stochastic processes

+ 1 Post Doc +1 AGPR+ 4 PhD

Random matrices

a_{ij} random, N large.

$$A_N = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{2N} \\ \vdots & \dots & \ddots & \dots & \vdots \\ \vdots & \dots & \ddots & \dots & \vdots \\ a_{N1} & \cdots & \cdots & \cdots & a_{NN} \end{pmatrix}$$

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How does the spectrum looks like when N goes to infinity ? What about the eigenvectors (localized or not)? Universality ? Non-normal matrices ? relation with operator algebra (and free probability) ?

Beta-ensembles

When A_N is Hermitian and the entries Gaussian, the joint law of the eigenvalues is given by

$$dQ_N^{\beta,V}(\lambda) = \frac{1}{Z_N} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\beta N \sum V(\lambda_i)} \prod d\lambda_i$$

with $\beta = 1, 2, 4$ and $V = x^2/2$.

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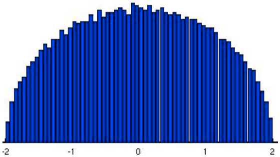
- (LLN) If V is continuous, going to infinity sufficiently fast, $\frac{1}{N} \sum \delta_{\lambda_i}$ converges towards the equilibrium measure μ_V



- (CLT)[Johansson 97, Shcherbina, G-Borot 11] Under more assumptions [cf 1 cut, off-critical], for smooth f,

$$\sum_{i=1}^{N} f(\lambda_i) - N \int f(x) d\mu_V(x) \to N(m_f, \sigma_f)$$

Local fluctuations of Beta ensembles

How does the spectrum look like when N goes to infinity and we look at detailed information like the behaviour of spacings $N(\lambda_i - \lambda_{i-1})$ or largest eigenvalue $\max_i \lambda_i$?

When $\beta = 2$, the law $Q_N^{2,V}$ is determinantal: its density is the square of a determinant

$$\prod_{i<j} |\lambda_i - \lambda_j| = \det(\lambda_j^i)$$

so that its local fluctuations can be analyzed by orthogonal polynomial techniques [Mehta 91', Tracy-Widom 94'].

Beta-ensembles: local fluctuations at the edge

Dumitriu-Edelman 02': Take $V(x) = \beta x^2/2$. Then $Q_N^{\beta,\beta x^2/2}$ is the law of the eigenvalues of

$$H_N^\beta = \begin{pmatrix} Y_1^\beta & \xi_1 & 0 & \cdots & 0 \\ \xi_1 & Y_2^\beta & \xi_2 & 0 & \vdots \\ 0 & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_{N-1} & Y_N^\beta \end{pmatrix}$$

where ξ_i are iid $N(0,1)$ and $Y_i^\beta \simeq \chi_{i\beta}$ independent.

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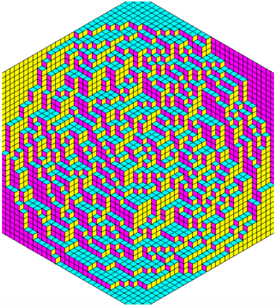
Bourgade-Erdòs-Yau 11', Shcherbina 13', Bekerman-Figalli-G 13': Universality: This remains true for general potentials provided off-criticality holds.

Random tiling in the hexagon

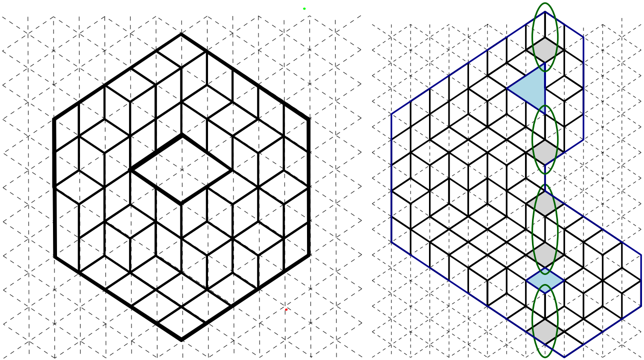
Take a tiling of the hexagon by lozenges uniformly at random



The distribution of horizontal tiles $\ell_1 < \ell_2 < \cdots < \ell_N$ along a vertical line is proportionnal to

$$\prod_{i<j} |\ell_i - \ell_j|^2 w(\ell_i)$$

Random tiling in domains constructed by gluing trapezoid



The distribution of horizontal tiles $\ell_1 < \ell_2 < \cdots < \ell_N$ along a vertical line is proportionnal to

$$\prod_{i<j} |\ell_i - \ell_j|^{\theta_{i,j}} w(\ell_i)$$

with $\theta_{i,j} \in \{0, 1, 2\}$.

Discrete β-ensembles ($\beta = 2\theta$)

For configurations ℓ such that $\ell_{i+1} - \ell_i - \theta \in \mathbb{N}$, $\ell_i \in [aN, bN]$, it is given by:

$$P_N^{\theta,w}(\ell) = \frac{1}{Z_N^{\theta,w}} \prod_{1\le i<j\le N} I_\theta(\ell_j, \ell_i) \prod w(\ell_i),$$

$$\text{where } I_\theta(\ell', \ell) = \frac{\Gamma(\ell' - \ell + 1)\Gamma(\ell' - \ell + \theta)}{\Gamma(\ell' - \ell)\Gamma(\ell' - \ell + 1 - \theta)}$$

Note that $I_\theta(\ell', \ell) \simeq |\ell' - \ell|^{2\theta}$ with $=$ if $\theta = 1, 1/2$.

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We can study the convergence, global fluctuations of the empirical measures

$$\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\ell_i / N}$$

and fluctuations of the extreme particles of the liquid region [Borodin, Borot, Gorin, G., Huang]

Convergence of the empirical measure

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Theorem

Assume that $w(\ell) \simeq e^{-NV(\ell/N)}$ *with* V *continuous on* $[a, b]$. *Then* $\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^N \delta_{\ell_i/N}$ *converges almost surely towards* μ_V *which minimizes*

$$\mathcal{E}(\mu) = \int V(x) d\mu(x) - \theta \int \int \ln |x - y| d\mu(x) d\mu(y)$$

over probability measures with density bounded by $1/\theta$.

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Proof

$$P_N^{\theta,w}(\ell) \simeq \frac{1}{Z_N^{\theta,w}} e^{-N^2 \mathcal{E}(\hat{\mu}_N)}, \quad \theta \#\{i : \ell_i/N \in [\alpha, \beta]\} \le N(\beta - \alpha) + 1$$

Fluctuations of the largest particles

For configurations ℓ such that $\ell_{i+1} - \ell_i - \theta \in \mathbb{N}$, $\ell_i \in [aN, bN]$,

$$P_N^{\theta,w}(\ell) = \frac{1}{Z_N^{\theta,w}} \prod_{1 \le i < j \le N} I_\theta(\ell_j, \ell_i) \prod w(\ell_i),$$

Theorem (Huang-G 17')

Under technical assumptions [one cut, off-criticality, analyticity], the largest particle ℓ_N fluctuates according to the Tracy-Widom 2θ distribution:

$$\lim_{N\to\infty} P_N^{\theta,w}\left(N^{-1/3}(\ell_N - N\beta) \ge t\right) = F_{2\theta}(t)$$

if $\beta = \min\{t : \mu_V((-\infty, t))\} = 1$.

Idea of the proof

- Rigidity (cf Erdos, Schlein, Yau 06'): for any $a > 0$

$$P_N^{\theta,w}(\sup_i |\ell_i - N\gamma_i| \geq \frac{N^a}{\min\{i/N, 1 - i/N\}^{1/3}}) \leq e^{-(\log N)^2}$$

where $\mu_V((-\infty, \gamma_i)) = i/N$.

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- One can compare the law of the extreme particles, at distance of order $N^{1/3} \gg 1$ (the mesh of the tiling) with the law of the extreme particles for the continuous model and deduce the 2θ-Tracy-Widom fluctuations.

Rigidity and Nekrasov equations

- Rigidity is obtained by proving that the Stieljes transform

$$G_N(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{z - \ell_i / N}$$

  is close to its deterministic limit for $\Im z \geq N^{-1+\delta}$. This is enough to show that the number of particles in an interval I of size $N^{-1+2\delta}$ is approximately $N\mu_V(I)$.
- Estimating the Stieljes equations is done thanks to the analysis of equations, analogous to loop or Dyson-Schwinger equations, derived by Nekrasov for the correlators (all moments of G_N), concentration of measures, and multiscale analysis.

Related questions and problems

- Several cuts (JW G. Borot and V. Gorin)
- Fluctuations of the surface of random tilings (Bufetov, Gorin)
- More general interactions (cf JW Borot and Kozlowski on sinsh model)
- Higher dimensions (cf Leblé-Serfaty)
- Pb: Universality: results are still restricted to very specific interactions (unknown for exact Coulomb gas in the discrete setting or Gamma interaction in the continuous).
- Fluctuations in the bulk ?
- Integrable systems?