Hilbert metric, singular elliptic PDEs and gas dynamics

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Team *PDEs and Modelling*

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Main themes

Project INRIA - NUMED. Complex math. modelling in Medecine: brain stroke, cancer. Multiscale analysis. Wave propagation in complex media. Statistical parameter estimates for PDEs. Numerical simulations.

Collaborations with medical teams.

Fluid dynamics. Hydrodynamics. Boundary layers. Statistical Physics. Boltzmann & kinetic equations. Mathematical modelling. Relative entropy method.

The equation ; 1st formulation

Unknown $\phi : \Omega \to \mathbb{R}$; planar bounded domain.

$$\operatorname{div} \frac{\nabla\phi}{\sqrt{2\phi + |\nabla\phi|^2}} + \frac{2}{\sqrt{2\phi + |\nabla\phi|^2}} = 0. \tag{1}$$

In quasilinear form

$$(2\phi + |\nabla\phi|^2)\Delta\phi - D^2\phi(\nabla\phi, \nabla\phi) + 4\phi + |\nabla\phi|^2 = 0.$$

The equation ; 2nd formulation

Auxiliary unknown $w = \sqrt{2\phi}$ (when $\phi > 0$)

$$\operatorname{div} \frac{\nabla w}{\sqrt{1+|\nabla w|^2}} + \frac{2}{w\sqrt{1+|\nabla w|^2}} = 0. \tag{2}$$

or

$$(1+|\nabla w|^2)\Delta w - \mathrm{D}^2 w(\nabla w, \nabla w) + 2\,\frac{1+|\nabla w|^2}{w} = 0.$$

Motivation (I)

The PDE is the Euler-Lagrange equation of either

$$L[\phi] = \int \sqrt{2\phi + |\nabla \phi|^2}\, \frac{dx}{\phi^{3/2}} \qquad \text{or} \qquad \mathcal{L}[w] = \int \sqrt{1 + |\nabla w|^2}\, \frac{dx}{w^2}.$$

Interpretation :

The graph $x_3 = w(x_1, x_2)$ is a non-parametric minimal surface in the Poincaré's half-space

$$\mathbb{H}^3 = \{x \in \mathbb{R}^3 \,|\, x_3 > 0\}, \qquad ds^2 = \frac{1}{x_3^2}(dx_1^2 + dx_2^2 + dx_3^2),$$

of constant *negative* curvature.

Motivation (II)

The PDE can be recast in terms of principal curvatures of a rotationally symmetric hypersurface in $\mathbb{R}^4$ as

$$\kappa_3 = \frac{1}{2}(\kappa_1 + \kappa_2).$$

Again, a parametric equation

$$\sqrt{x_4^2 + x_3^2} = w(x_1, x_2).$$

Motivation (III)

Euler equations for isentropic gas dynamics

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) &= 0, \\ \partial_t (\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) &= 0. \end{aligned}$$

For self-similar flows (*Riemann Problem*)

$$\rho = \rho\left(\frac{x}{t}\right), \qquad \mathbf{u} = \mathbf{u}\left(\frac{x}{t}\right),$$

the system becomes

$$\operatorname{div}(\rho \mathbf{v}) + 2\rho = 0, \qquad (\mathbf{v} \cdot \nabla)\mathbf{v} + \mathbf{v} + \frac{1}{\rho} \nabla p = 0,$$

where $\mathbf{v}(y) := \mathbf{u}(y) - y$ is the *pseudo-velocity*.

Sub-case : the potential flow

$$\mathbf{v} = \nabla \phi$$

satisfies Bernoulli's Theorem ($\imath$ the *enthalpy*)

$$\frac{1}{2}|\nabla\phi|^2 + \phi + \imath(\rho) = 0, \qquad \imath'(\rho) = \frac{1}{\rho}p'(\rho),$$

that is

$$\rho = h(2\phi + |\nabla\phi|^2).$$

Whence a 2^{nd}-order PDE

$$\operatorname{div}\left(h(2\phi + |\nabla\phi|^2)\nabla\phi\right) + 2h(2\phi + |\nabla\phi|^2) = 0. \tag{3}$$

The *Chaplygin / von Kármán* equation of state leads to

$$h(s) = s^{-1/2}.$$

Then (3) reduces to our PDE.

Type of the equation

- Hyperbolic type if $\phi < 0$

(think to the *wave equation*).

- Elliptic type if $\phi > 0$

(think to the *Laplace equation*).

- Degeneracy when ϕ vanishes.

We look for positive solution. Hence we may work in terms of $w = \sqrt{2\phi}$.

The boundary condition

$$\phi|_{\partial\Omega} = 0 \qquad (\text{i.e.} \qquad w|_{\partial\Omega} = 0). \tag{4}$$

Interpretations

(I) The minimal surface is *complete,*

M. Anderson (*Inventiones Math.* 1982) proved the existence of parametric complete minimal surface, using geometric measure theory.

He considers also the non-parametric case, but then his “proof” has serious flaws.

(II) The surface of revolution has no boundary,

(III) The domain Ω is the subsonic zone of a flow.

The boundary $\partial\Omega$ is calculated *a priori* from the Riemann data, by solving plnar wave interactions.

Summing up, our BVP is

$$\begin{aligned}
(2\phi+|\nabla\phi|^2)\Delta\phi - D^2\phi(\nabla\phi,\nabla\phi)+4\phi+|\nabla\phi|^2 &= 0 \quad \text{in } \Omega,\\
\phi &> 0 \quad \text{in } \Omega,\\
\phi &= 0 \quad \text{on } \partial\Omega.
\end{aligned}$$

The equation is elliptic in Ω ; it degenerates along the boundary.

At $x \in \partial\Omega$ the ellipticity is lost. Its failure occurs in the *tangent* direction (unlike Tricomi equation) : a non generic degeneracy (of *Keldysh* type).

But then,

Where is the data ?

The PDE is homogeneous, as well as the boundary condition ...

The data of the problem is the domain Ω !

For the Chaplygin gas, only the supersonic part of the flow is not explicit.

A necessary condition for existence

Non generic degeneracy is associated with an explicit formula of every derivative at the boundary $\partial\Omega$!

For instance

$$\kappa\frac{\partial\phi}{\partial\nu}=-1,\qquad (\kappa \text{ the curvature}).$$

With $\phi>0$ in Ω and $\phi=0$ on $\partial\Omega$, this yields:

Proposition 1 *A necessary condition for the existence of a solution, of class $\mathcal{C}^2$ up to the boundary, is that Ω be uniformly convex :*

$$\inf\kappa>0.$$

♠

The convexity is sufficient !

Theorem 1 (DS, 2009, 2015.) *Let Ω be a bounded planar convex domain.*

Then there exists a unique positive solution to the BVP,

$$\phi \in \mathcal{C}^{\infty}(\Omega) \cap \mathcal{C}(\overline{\Omega}).$$

♡

Difficulties : – non-linearity, – degeneracy at the boundary, – non-uniform ellipticity (unless $|\nabla w|$ is bounded, unlikely).

Maximum Principle is the only tool ...

The Maximum Principle

The equation writes $N(w, \nabla w, \mathrm{D}^2 w) = 0$ with

- (ellipticity) $S \mapsto N(w, p, S)$ strictly increasing in $\mathbf{Sym}_2(\mathbb{R})$,
- $w \mapsto N(w, p, S)$ decreasing in $(0, +\infty)$.

MP 1 *If w_+ is a super-solution ($N(w_+, \nabla w_+, \mathrm{D}^2 w_+) \leq 0$ in ω), if w_- is a sub-solution ($N(w_-, \nabla w_-, \mathrm{D}^2 w_-) \geq 0$), and if $w_- \leq w_+$ over $\partial\omega$, then*

$$w_- \leq w_+ \quad \textit{dans } \omega.$$

Strategy

1. The MP implies uniqueness.

2. Comparison with exact solutions $\phi^{r,m}(x) = \frac{1}{2}(r^2 - |x-m|^2)$ yield an explicit lower bound $w_{\min}$ satisfying

$$w_{\min} > 0 \text{ in } \Omega, \qquad w_{\min} = 0 \text{ over } \partial\Omega.$$

3. Similiar idea, with a little more calculations, yields an explicit upper bound $w_{\max}$ such that

$$w_{\max} > 0 \text{ in } \Omega, \qquad w_{\max} = 0 \text{ over } \partial\Omega.$$

Not the end of the story, because of the lack of uniform ellipticity.

4. Establish a Lipschitz estimate, to ensure the uniform ellipticity.

The most delicate point, since it is known to be only local !

5. Apply the regularity theory (*cf* Gilbarg–Trudinger) to get an estimate of $\mathrm{D}^2 w$ (or $\mathrm{D}^2\phi$ as well). This ensure the pre-compactness for $\nabla\phi$.

6. Proceed with a continuation method, with respect to the parameters

$$\operatorname{div}\frac{\nabla\phi}{\sqrt{2\phi+|\nabla\phi|^2}}+\frac{\mu}{\sqrt{2\phi+|\nabla\phi|^2}}=0, \qquad \mu\in[0,2],$$

and

$$\phi|_{\partial\Omega}=\varepsilon>0, \qquad \varepsilon\to 0^+,$$

(for $\mu=0$, the solution is $\phi\equiv\varepsilon$).

The Lipschitz estimate

Along $\partial\Omega$,

$$|\nabla\phi| = \frac{1}{\kappa}, \qquad |\nabla w| = \infty.$$

Lemma 1 (2009) *If* $\min_{\partial\Omega}\kappa > 0$ *(uniformly convex domain), the Lipschitz estimate of* ϕ *at the boundary + a (delicate) MP yields* a priori

$$\|\nabla\phi\|_\infty < C(\min\kappa, \mathrm{diam}\Omega).$$

$\diamondsuit$

Useless when κ vanishes (flat points) ; one even has $\|\nabla\phi\|_\infty = \infty$.

Instead, let us use the *invariance* of the PDE

under translation : if w is a solution in Ω, then $x \mapsto w(x+a)$ is a solution in $\Omega - a$,

under rescaling : likewise, $x \mapsto \lambda w(\lambda^{-1}x)$ is a solution in $\lambda\Omega$.

For $a, b \in \Omega$, define

$$p(a|b) = \inf\{\lambda > 0 \,|\, \Omega - b \subset \lambda(\Omega - a)\} \in [1, +\infty).$$

For $\lambda \geq p(a|b)$,

$$z(x) := \lambda w\left(\frac{x}{\lambda} + a\right)$$

is a solution of the BVP in $\lambda(\Omega - a)$. It is thus a super-solution in $\Omega - b$. Therefore the MP gives

$$w(x+b) \leq z(x), \qquad \forall x \in \Omega - b.$$

Put $x = 0$:

$$w(b) \leq w(a)p(a|b).$$

Lipschitz estimate :

$$|\log w(a) - \log w(b)| \leq d_\Omega(a,b) := \max\{\log p(a|b), \log p(b|a)\},$$

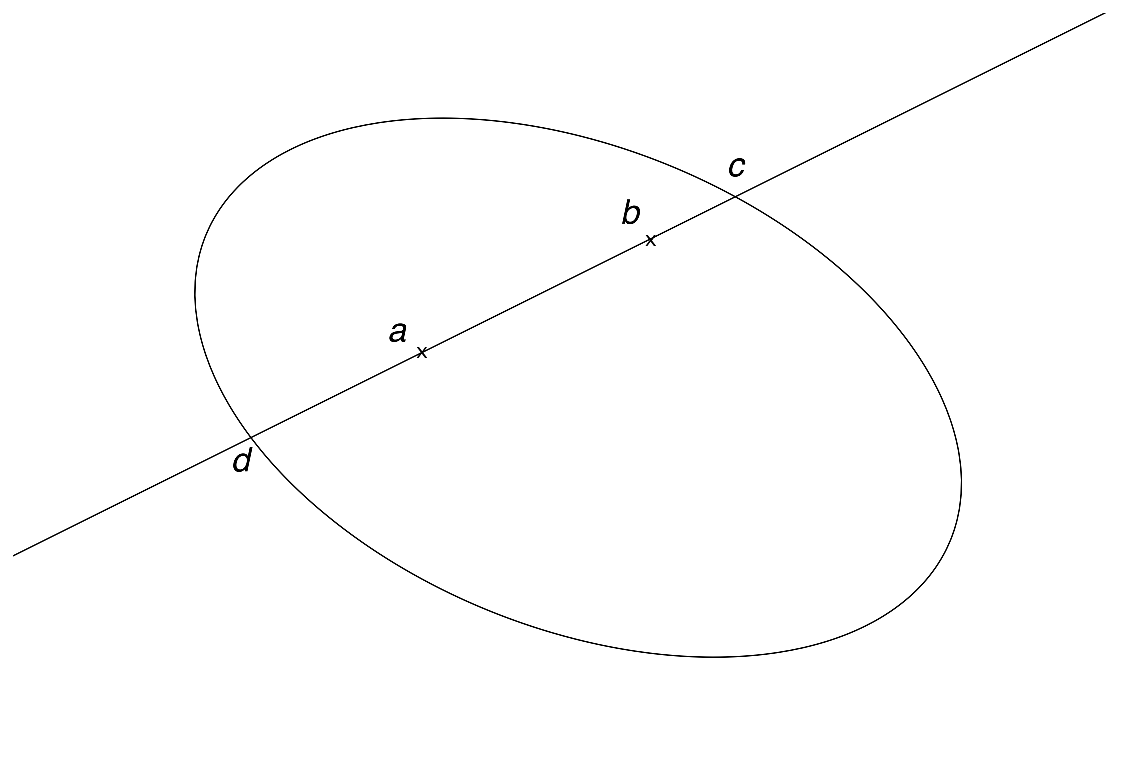
where d_Ω is the *Thompson* distance over Ω.

Equivalence $\frac{1}{2}d_H \leq d_\Omega \leq d_H$, where

$$d_H(a,b) := \log p(a|b) + \log p(b|a).$$

d_H is the Hilbert metric in Ω.

$$d_H(a,b) = \log \frac{\overline{ac} \cdot \overline{bd}}{\overline{ad} \cdot \overline{bc}}.$$



Corollary 1 *Let w be the unique solution w of the BVP. Then $\log w$ is 1-Lipschitz with respect to the Hilbert metric.*

$\diamondsuit$

Summary

- The solution is *a priori* bounded above and below by $w_{\max} > w_{\min} > 0$, both vanishing at the boundary,

- On every compact sub-domain, $|\nabla w|$ is *a priori* bounded (d_H is locally equivalent to the Euclidian metric),

- Thus the PDE has uniform ellipticity, away from the boundary,

- Therefore the regularity theory for elliptic equations applies. Whence *a priori* bounds for $\mathrm{D}^2 w$ on compact sub-domains,

- By Ascoli–Arzela, one has pre-compactness of (approximate) solutions in $\mathcal{C}^1$. The passage to the limit in the equation is valid.

- The limit of approximate solutions is a solution.

- By the MP, the solution is unique.

Up to technical details, this proves Theorem 1.

Thanks for your attention !