On the decidability of the word problem for amalgamated free products of inverse semigroups

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Abstract

We study inverse semigroup amalgams $[S_1, S_2; U]$, where S_1 and S_2 are finitely presented inverse semigroups with decidable word problem and U is an inverse semigroup with decidable membership problem in S_1 and S_2. We use a modified version of Bennett's work on the structure of Schützenberger graphs of the $\mathcal{R}$-classes of $S_1 *_U S_2$ to state sufficient conditions for the amalgamated free products $S_1 *_U S_2$ having decidable word problem.

Key words: inverse semigroups, presentation, amalgamated free products, Schützenberger automata.

1. Introduction

A semigroup S is *regular* when for each $s \in S$ there exists $t \in S$ (called an *inverse* of s) such that $s = sts$ and $t = tst$. If each $s \in S$ has a unique inverse (denoted by s^{-1}) then S is called an *inverse semigroup*. For any inverse semigroup S, the *natural partial order* $\leq$ is defined by $u \leq v$ if and only if $u \in \mathrm{E}_S v$, where E_S denotes the semilattice of idempotents of S. The set $[u\uparrow] = \{v \in S \mid u \leq v\}$ is called *order filter* of $u \in S$. We refer the reader to Petrich [12] for many other standard results and ideas about inverse semigroups.

If $\mathscr{C}$ is a category of semigroups, S_1 and S_2 are $\mathscr{C}$-semigroups such that $S_1 \cap S_2 = U$ is a non-empty $\mathscr{C}$-subsemigroup of both S_1 and S_2, the triple $\mathfrak{U} = [S_1, S_2; U]$ is called an *amalgam* of $\mathscr{C}$-semigroups $\{S_1, S_2\}$ with *core* U. The amalgam $\mathfrak{U}$ is said to be *strongly embeddable* in a $\mathscr{C}$-semigroup if there exist a $\mathscr{C}$-semigroup S and embeddings $\phi_i : S_i \hookrightarrow S$ such that $\phi_1|_U = \phi_2|_U$ and $\phi_1(S_1) \cap \phi_2(S_2) = \phi_1(U) = \phi_2(U)$. A semigroup amalgam is not necessarily (strongly) embeddable and a large literature is devoted to the (strong) embeddability of semigroup amalgam, we refer the reader to Howie [7] for some references on this problem. In this paper we will be concerned only with inverse semigroup amalgams, i.e. in amalgams $[S_1, S_2; U]$ where S_1, S_2 and U are inverse semigroups. A very important theorem of Hall [5] states that the category of inverse semigroups has the *Strong Amalgamation Property*. The *amalgamated free product* (or the *free product with amalgamation*) of S_1 and S_2, with *core* U, in the category of inverse semigroups, is denoted by $S_1 *_U S_2$. If $U = \varnothing$ we have the *free product* $S_1 * S_2$.

We briefly recall the notion of presentation of inverse semigroups (see Stephen [13] for more details). Given a finite alphabet X, let X^{-1} be a disjoint alphabet of formal inverses of X so that there exists an involutory one-to-one correspondence between X and X^{-1}, i.e. for each $x \in X$ there exists a unique $x^{-1} \in X^{-1}$ and $(x^{-1})^{-1} = x$. We denote with $\left(X \cup X^{-1}\right)^{+}$ $[\left(X \cup X^{-1}\right)^{*}]$ the *free semigroup* [*monoid*] *with involution* on X, whose elements are also called *words*. Let ρ be the *Vagner congruence* on $\left(X \cup X^{-1}\right)^{+}$ (we write ρ_X if X needs to be specified), the quotient semigroup $\mathrm{FIS}(X) = \left(X \cup X^{-1}\right)^{+}/\rho$ is the *free inverse semigroup* on X. The *free group* on X is denoted $\mathrm{FG}(X)$. For each $w \in \left(X \cup X^{-1}\right)^{*}$, $\mathrm{r}(w)$ denotes the *reduced form* (in the sense of $\mathrm{FG}(X)$) of w. It is well known that $\mathrm{FG}(X)$ can be represented via the set $\mathrm{r}\left(\left(X \cup X^{-1}\right)^{*}\right)$ of reduced words over the alphabet $\left(X \cup X^{-1}\right)$. Now let R be a binary relation on a semigroup, we denote with R^{e} $[R^{\mathrm{c}}]$ the *equivalence* [*congruence*] *generated by R*. A *presentation* of an inverse semigroup is a pair $(X; R)$, where R is a binary relation on $\left(X \cup X^{-1}\right)^{+}$. The inverse semigroup $S = \left(X \cup X^{-1}\right)^{+}/(R \cup \rho)^{\mathrm{c}}$ is said to be *presented* by the *generators* X and the *relation* R, and is denoted by $S = \mathrm{Inv}\,\langle X|R\rangle$. The fundamental question associated with the concept of a presentation is the decidability of the *word problem*, i.e. the existence of an effective algorithm that, for a given inverse semigroup $S = \mathrm{Inv}\,\langle X|R\rangle = \left(X \cup X^{-1}\right)^{+}/\tau$ with $\tau = (R \cup \rho)^{\mathrm{c}}$ and two arbitrary words $u, v \in \left(X \cup X^{-1}\right)^{+}$, decides whether or not $u\tau$ and $v\tau$ are the same element of S or not.

In this work all the automata and the underlying graphs that appear will be inverse automata and inverse graphs. An *X-inverse word graph* Γ is a strongly connected labeled digraph with a non-empty vertex set $\mathrm{V}(\Gamma)$, whose set of edges $\mathrm{E}(\Gamma)$ is labeled by elements of $\left(X \cup X^{-1}\right)$ and is involutive, i.e. $(v', x, v'') \in \mathrm{E}(\Gamma)$ if and only if $(v', x, v'')^{-1} = (v'', x^{-1}, v') \in \mathrm{E}(\Gamma)$. Recall that a graph is called *strongly connected* when for each pair of vertices v' and v'' there exists a $v' - v''$ path. An *X-inverse subgraph* Λ of Γ is an X-inverse word graph whose vertex and edge sets are subset of the respective sets of Γ, and we write $\Lambda \subseteq \Gamma$. Note that in particular Λ must be strongly connected. A *morphism* from the X-inverse word graph Γ to the X-inverse word graph Ω is a pair of maps $\phi = (\phi_{\mathrm{V}}, \phi_{\mathrm{E}})$, where $\phi_{\mathrm{V}} : \mathrm{V}(\Sigma) \to \mathrm{V}(\Omega)$ and $\phi_{\mathrm{E}} : \mathrm{E}(\Sigma) \to \mathrm{E}(\Omega)$, such that $\phi_{\mathrm{E}}(v', x, v'') = (\phi_{\mathrm{V}}(v'), x, \phi_{\mathrm{V}}(v''))$. Note that ϕ is completely determined by ϕ_{V}. In the following we will use the same symbol ϕ for both ϕ_{V} and ϕ_{E}. An *X-inverse word automaton* is a triple $\mathcal{A} = (v', \Gamma, v'')$, where v' and v'' are vertices of Γ, and are called starting and terminal vertices respectively. The *language* $\mathrm{Lang}[\mathcal{A}] \subseteq \left(X \cup X^{-1}\right)^{*}$ is the set of words that label a $v' - v''$ path on Γ. A *morphism* from the X-inverse automaton $\mathcal{A}$ to the X-inverse automaton $\mathcal{A}'$ is a morphism of the underlying graphs that send the starting and terminal vertices of $\mathcal{A}$ into the starting and terminal vertices of $\mathcal{A}'$ respectively. It is easy to see that the X-inverse word graphs [resp. X-inverse word automaton] with their morphisms form a category.

Given an equivalence relation μ on the vertex set of the X-inverse automaton $\mathcal{A} = (v', \Gamma, v'')$, the *quotient graph* Γ/μ has vertex set $\mathrm{V}(\Gamma/\mu) = \mathrm{V}(\Gamma)/\mu$ and edge set induced by the quotient projection in the obvious way. The *quotient automaton* is the X-inverse automaton $\mathcal{A}/\mu = (v'\mu, \Gamma/\mu, v''\mu)$. An X-inverse word graph Γ

is *deterministic* when $(w,x,w'),(w,x,w'') \in \mathrm{E}(\Gamma)$ implies $w' = w''$. Given an X-inverse word graph Γ it is easy to prove that there is a minimum equivalence μ_{det} on the vertices such that the quotient graph is deterministic. We call the quotient $\Gamma/\mu_{\mathrm{det}}$ the *determinized form* of Γ, and we denote it by Γ_{det}. Similarly we define the *determinized form* $\mathcal{A}_{\mathrm{det}}$ of an X-inverse automaton $\mathcal{A}$. See STEPHEN [13] for more details concerning the categories of X-inverse word graphs and X-inverse automata.

Central to all of this work is the notion of *Schützenberger automaton* $\mathcal{A}(X;R;s)$ of an element $s \in \mathrm{Inv}\,\langle X|R\rangle = \left(X \cup X^{-1}\right)^{+}/\tau$. The underlying graph of $\mathcal{A}(X;R;s)$ is the *Schützenberger graph* $\mathcal{S}\Gamma(X;R;s)$, whose vertex and edge sets are respectively

$$
\begin{aligned}
(1)\qquad & \mathrm{V}(\mathcal{S}\Gamma(X;R;s)) = \{v \in \mathrm{Inv}\,\langle X|R\rangle \mid v\mathcal{R}s\}\,,\\
& \mathrm{E}(\mathcal{S}\Gamma(X;R;s)) = \{(v_1,x,v_2) \mid v_1,v_2 \in \mathrm{V}(\mathcal{S}\Gamma(X;R;s)), v_2 = v_1 \cdot x\tau\}\,.
\end{aligned}
$$

where $(v_1,x,v_2) \in \mathrm{E}(\mathcal{S}\Gamma(X;R;s))$ denotes the edge whose starting and terminal vertices are respectively v_1 and v_2, and $x \in \left(X \cup X^{-1}\right)$ is the label of the edge. The relation $\mathcal{R}$ in (1) is as usual the *right Green's relation*. We recall that $s\mathcal{R}t$ if and only if $ss^{-1} = tt^{-1}$. The main property of Schützenberger automata is that $u\tau = v\tau$ if and only if $\mathcal{A}(X;R;u\tau) = \mathcal{A}(X;R;v\tau)$, so these automata can be employed to study the word problem in $\mathrm{Inv}\,\langle X|R\rangle$.

In [13] STEPHEN describes an iterative procedure to "build" a Schützenberger automaton $\mathcal{A}(X;R;s)$ from a sequence $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ of approximate automata. We recall that, given a word $w \in \left(X \cup X^{-1}\right)^{+}$, an automaton $\mathcal{A}$ is an *approximate automaton* for the Schützenberger automaton $\mathcal{A}(X;R;w\tau)$, written $\mathcal{A} \rightsquigarrow \mathcal{A}(X;R;w\tau)$, when the language $\mathrm{Lang}[\mathcal{A}]$ is contained in the language $\mathrm{Lang}[\mathcal{A}(X;R;w\tau)]$ of the second and $w'\tau = w\tau$ for some $w' \in \mathrm{Lang}[\mathcal{A}]$. More precisely the *Stephen iterative procedure* sets up a direct system (of approximate automata) in the category $\mathscr{A}_X$ of X-inverse automata whose direct limit is the Schützenberger automaton $\mathcal{A}(X;R;w\tau)$. We refer the reader to the original work of STEPHEN [13] for all needed details.

This paper developes sufficient conditions for the decidability of the word problem for the amalgamated free product $S_1 *_U S_2$ for a given inverse semigroup amalgam $\mathfrak{U} = [S_1,S_2;U]$ where S_1 and S_2 have given presentations and decidable word problem.

In [3] BIRGET, MARGOLIS and MEAKIN showed that the word problem for a *generic* semigroup amalgam $\mathfrak{U} = [S_1,S_2;U]$ is in general undecidable even if S_1 and S_2 have given presentations with decidable word problem, and U is a free semigroup which is unitary in each S_i and has decidable membership problem for S_1 and S_2. Thus the situation is very different from the situation in group amalgams $\mathfrak{V} = [G_1,G_2;V]$, where by the *normal form theorem* (see e.g. LYNDON and SCHUPP [9]) the word problem is decidable if G_1 and G_2 have decidable word problem and the core V has decidable membership problem in each group.

In [1, 2] BENNETT introduced the class of *lower bounded* inverse semigroup amalgams and developed an algorithm for setting up direct systems in the category $\mathscr{G}_X$ of X-inverse word graphs whose direct limits are the Schützenberger graphs of amalgamated free products of lower bounded amalgams. Using his results, CHERUBINI, MEAKIN and PIOCHI [4] proved the decidability of the word problem for amalgamated free products of the form $\mathrm{FIS}(X_1) *_U \mathrm{FIS}(X_2)$, where the inverse semigroup U is finitely generated.

In this work we will generalize this result by considering inverse semigroups S_1 and S_2 that are not free, in which case the Schützenberger graphs of the $\mathcal{R}$-classes of S_1 and S_2 may be infinite. To overcome this difficulty we make use of appropriate approximate Munn trees: given two arbitrary words w and z, we provide a construction that builds a Munn tree $\mathrm{MT}(\widehat{w}_1)$ of a word $\widehat{w}_1$ such that $\widehat{w}_1$ and w represent the same element of $S_1 *_U S_2$, then we build up a sequence of Munn tree $\{\mathrm{MT}(\widehat{w}_k)\}_k$ such that reading z by the Schützenberger automaton of the element of $S_1 *_U S_2$ represented by w is simulated using the Munn trees $\{\mathrm{MT}(\widehat{w}_k)\}_{k\in\{1,\ldots,K\}}$, where $K \in \mathbb{N}^+$ is a computable integer.

2. Background

We briefly recall the "shape" of the Schützenberger automata for the free inverse semigroup case. Consider $\mathrm{FIS}(X) = \left(X \cup X^{-1}\right)^+/\rho = \mathrm{Inv}\left\langle X|\varnothing\right\rangle$ and a word $w = w_1 w_2 \ldots w_{|w|} \in \left(X \cup X^{-1}\right)^+$ where $w_i \in \left(X \cup X^{-1}\right)$. We call *prefix set* of w the following set of words

$$\mathrm{pref}(w) = \left\{\varepsilon,\, w_1,\, w_1w_2,\, \ldots,\, w_1w_2\ldots w_{|w|}\right\},$$

where ε denotes the empty word. The *Munn Tree* of the word w is the X-inverse word graph $\mathrm{MT}_X(w)$ having vertex and edge sets respectively[1]

$$\begin{aligned} \mathrm{V}(\mathrm{MT}_X(w)) &= \{\mathrm{r}(v) \,|\, v \in \mathrm{pref}(w)\} = \mathrm{r}\left(\mathrm{pref}(w)\right), \\ \mathrm{E}(\mathrm{MT}_X(w)) &= \left\{(v, x, \mathrm{r}(vx)) \,|\, v, \mathrm{r}(vx) \in \mathrm{r}\left(\mathrm{pref}(w)\right),\, x \in \left(X \cup X^{-1}\right)\right\}. \end{aligned}$$

There is an isomorphism between the Schützenberger automata $\mathcal{A}(X; \varnothing; w\rho)$ and the *Munn automata* $(\varepsilon, \mathrm{MT}_X(w), \mathrm{r}(w))$ sending the vertices

$$(ww^{-1})\rho, w\rho \in \mathrm{V}(\mathcal{S}\Gamma(X; \varnothing; w\rho))$$

to the vertices

$$\varepsilon, \mathrm{r}(w) \in \mathrm{V}(\mathrm{MT}_X(w))$$

respectively.

Let $(X_1; R_1)$ and $(X_2; R_2)$ be totally disjoint presentations for the inverse semigroups

$$\begin{aligned} S_1 &= \mathrm{Inv}\left\langle X_1|R_1\right\rangle = \left(X_1 \cup X_1^{-1}\right)^+/\tau_1, \\ S_2 &= \mathrm{Inv}\left\langle X_2|R_2\right\rangle = \left(X_2 \cup X_2^{-1}\right)^+/\tau_2, \end{aligned}$$

[1] The given definition of Munn tree differs from the original definition on [11] only on the label of the vertices, but the other properties are preserved.

where $\tau_1 = (\rho_{X_1} \cup R_1)^{\mathrm{c}}$ and $\tau_2 = (\rho_{X_2} \cup R_2)^{\mathrm{c}}$. We also denote

$$X = X_1 \cup X_2, \quad X^{-1} = X_1^{-1} \cup X_2^{-1}, \quad R = R_1 \cup R_2.$$

Let U be an inverse semigroup isomorphic to a subsemigroup of S_1 and S_2, so that the triple $\mathfrak{U} = [S_1, S_2; U]$ forms an amalgam of inverse semigroups. To give a presentation for the amalgamated free product $S_1 *_U * S_2$ we can fix a pair of injective maps

$$\mathrm{w}_1 : U \to \left(X_1 \cup X_1^{-1}\right)^+, \quad \mathrm{w}_2 : U \to \left(X_2 \cup X_2^{-1}\right)^+,$$

such that $(\mathrm{w}_i(u))\tau_i = u$ for each $u \in U$. Then we can define the binary relation

$$W = \{(\mathrm{w}_1(u), \mathrm{w}_2(u)) \mid u \in U\},$$

and we obtain

$$S_1 * S_2 = \mathrm{Inv}\,\langle X | R\rangle = \left(X \cup X^{-1}\right)^+ / \tau,$$
$$S_1 *_U S_2 = \mathrm{Inv}\,\langle X | R \cup W\rangle = \left(X \cup X^{-1}\right)^+ / \eta,$$

where $\tau = (\rho_X \cup R)^{\mathrm{c}}$ and $\eta = (\rho_X \cup R \cup W)^{\mathrm{c}} = (\tau \cup W)^{\mathrm{c}}$. Since $(X_1; R_1)$ and $(X_2; R_2)$ are totally disjoint no confusion arises denoting by "$\leq$" the natural partial order in S_1, S_2, $S_1 * S_2$ and $S_1 *_U S_2$. We define the *order filter* of an element $s \in S_1 \cup S_2$ *restricted* to U as the set[2]

$$[s{\uparrow}_U] = \{u \in U \mid s \leq u\}.$$

If $[s{\uparrow}_U]$ admits a minimum we denote it by $\mathrm{f}(s)$, i.e. $\mathrm{f}(s) \in [s{\uparrow}_U]$ and $\mathrm{f}(s) \leq u$ for all $u \in [s{\uparrow}_U]$. It's easy to prove that $\mathrm{f}(e) \in \mathrm{E}_U$, if it exists, for each $e \in \mathrm{E}_{S_i}$.

Given an X-inverse word graph Γ we call *lobe coloured by* i each maximal subgraph Δ of Γ whose edge labels are in X_i for a particular $i \in \{1, 2\}$. If $v \in \mathrm{V}(\Gamma)$ is a vertex of two distinct lobes, it is called an *intersection vertex*.

The set of all the intersection vertices of Γ is denoted by $\mathrm{IV}(\Gamma)$ and, for $v \in \mathrm{IV}(\Gamma)$, $\Delta_1(v)$ and $\Delta_2(v)$ denote the *adjacent* lobes coloured respectively by 1 and 2. The *lobe graph* of Γ is the graph LG_Γ (in the category $\mathscr{G}$ of Serre's graphs) given by

$$\begin{aligned}
\mathrm{V}(\mathrm{LG}_\Gamma) =& \{\Delta \subseteq \Gamma \mid \Delta \text{ is a lobe}\},\\
\mathrm{E}(\mathrm{LG}_\Gamma) =& \{(\Delta, \Delta') \in \mathrm{V}(\mathrm{LG}_\Gamma) \times \mathrm{V}(\mathrm{LG}_\Gamma) \mid \\
& \quad \Delta = \Delta_i(v),\ \Delta' = \Delta_{3-i}(v) \text{ for some } v \in \mathrm{IV}(\Gamma),\ i \in \{1, 2\}\}.
\end{aligned}$$

A graph Γ is called *cactoid* when LG_Γ is a finite tree and each pair of adjacent lobes has only one intersection vertex. Let Δ be a lobe coloured by $i \in \{1, 2\}$ of an X-inverse word graph Γ such that Δ is isomorphic to a Schützenberger graph $\mathcal{S}\Gamma(X_i; R_i; s_i)$, then for $v \in \mathrm{V}(\Delta)$ we denote by $\mathrm{e}_i(v)$ the idempotent in E_{S_i} such that $(v, \Delta, v) \simeq (v, \mathcal{S}\Gamma(X_i; R_i; s_i), v) \simeq \mathcal{A}(X_i; R_i; \mathrm{e}_i(v))$.

An inverse semigroup amalgam $\mathfrak{U} = [S_1, S_2; U]$ is *lower bounded* when it satisfy the following two conditions:

[2]We prefer the suggestive notation $[s{\uparrow}_U]$ instead of the Bennett notation $U_i(s)$ (where $s \in S_i$) used in [1].

(LB1) $[e{\uparrow}_U] = \varnothing$ or $\mathrm{f}(e) \in \mathrm{E}_U$ exists for each $i \in \{1,2\}$ and for each $e \in \mathrm{E}_{S_i}$;

(LB2) for each $i \in \{1,2\}$ and for each $e \in \mathrm{E}_{S_i}$, if $\{u_k\}_k$ is a sequence of idempotents in E_U such that $u_{k+1} \neq u_k$ and $u_{k+1} \leq \mathrm{f}(eu_k) \leq u_k$, then the sequence $\{u_k\}_k$ is finite.

It is useful for the sequel to provide a brief description of Bennett procedure that "build" the Schützenberger automaton $\mathcal{A}(X; R \cup W; w\eta)$, where $S_1 *_U S_2 = \mathrm{Inv}\,\langle X | R \cup W\rangle = \left(X \cup X^{-1}\right)^+ / \eta$ is the free product of a lower bounded amalgam. Bennett procedure consists in five constructions, applied starting from the approximate Munn automaton $\mathcal{A}_0 = (\alpha_0; \Gamma_0; \beta_0) = (\varepsilon, \mathrm{MT}_X(w), \mathrm{r}(w))$. Each construction is iterated until it cannot be applied anymore.

Bennett Construction A ([2, construction 1.10] or [4, step 1]). Starting from the approximate X-inverse automaton

$$\mathcal{A}_k = (\alpha_k, \Gamma_k, \beta_k) \rightsquigarrow \mathcal{A}(X; R; w\tau)$$

let $\Delta_k \subseteq \Gamma_k$ be a lobe coloured by $i \in \{1,2\}$ that is not $(X_i; R_i)$-closed. For $v_k \in \mathrm{V}(\Delta_k)$, a unique idempotent $e_k \in E_{S_i}$ exists such that $(v_k, \Delta_k, v_k) \rightsquigarrow \mathcal{A}(X_i; R_i; e_k)$. Then $(X_i; R_i)$-close the lobe Δ_k, obtaining the approximate X-inverse automaton $\mathcal{A}_{k+1} = (\alpha_{k+1}, \Gamma_{k+1}, \beta_{k+1})$, where[3]

$$\Gamma_{k+1} = \left(\left(\Gamma_k \amalg \mathcal{S}\Gamma(X_i; R_i; e_k)\right)/\mu\right)_{\mathrm{det}}, \qquad \text{with } \mu = \{((v_k, 1), (e_k, 2))\}$$

and $\alpha_{k+1}, \beta_{k+1}$ are the natural images of α_k, β_k. ■

Bennett Construction B ([2, construction 2.1] or [4, step 2]). Given the X-inverse word graph Γ_k such that

$$\Gamma_k \simeq \mathcal{S}\Gamma(X; R; s_k)$$

for some $s_k \in S_1 * S_2$, let $v \in \mathrm{IV}(\Gamma)$ be an intersection vertex $v \in \mathrm{IV}(\Gamma)$ such that $[\mathrm{e}_1(v){\Uparrow}_U] \neq [\mathrm{e}_2(v){\Uparrow}_U]$. Let $i \in \{1,2\}$ and $j = 3 - i$ such that

$$\varnothing \neq [\mathrm{e}_i(v){\Uparrow}_U] \nsubseteq [\mathrm{e}_j(v){\Uparrow}_U].$$

Assign $f = \mathrm{f}(\mathrm{e}_i(v))$ and build the X-inverse word graph Γ'_k as

$$\Gamma'_k = \left(\left(\Gamma_k \amalg \mathcal{S}\Gamma(X_j; R_j; f)\right)/\zeta\right)_{\mathrm{det}}, \qquad \text{with } \zeta = \{((v,1),(f,2))\}^{\mathrm{e}}.$$

and if needed reiterate Bennett construction A starting to obtain the $(X; R)$-closed X-inverse word graph Γ_{k+1}. ■

[3] We make use of the following notion of *disjoint union*: given a family of sets $\{A_k \mid k \in \{1,...,n\}\}$ we define their disjoint union as the set $\coprod_{k=1}^n A_k = \bigcup_{k=1}^n (A_k \times \{k\})$. If $\{\Gamma_k \mid k \in \{1,...,n\}\}$ is a family of X-inverse word graphs we define their disjoint union as the X-inverse word graph $\Gamma = \coprod_{k=1}^n \Gamma_k$ having vertex and edge set respectively

$$\mathrm{V}(\Gamma) = \coprod_{k=1}^n \mathrm{V}(\Gamma_k), \qquad \mathrm{E}(\Gamma) = \{((v_1,k), x, (v_2,k)) \mid (v_1, x, v_2) \in \mathrm{E}(\Gamma_k)\}.$$

An X-inverse automaton $\mathcal{A} = (\alpha, \Gamma, \beta)$ (or the underlying X-inverse word graph Γ) has the *lower bound equality property* **(LBE)** when $[\mathrm{e}_1(v)\!\upharpoonright_U] = [\mathrm{e}_2(v)\!\upharpoonright_U]$ for each $v \in \mathrm{IV}(\Gamma)$. Thus at the end of iterations of Bennett construction B the resulting X-inverse word graph satisfies **(LBE)**.

An X-inverse word graph Γ with **(LBE)** property has the *related pair separation property* **(RPS)** when there is no $u \in U$ such that $\mathrm{w}_1(u)$ or $\mathrm{w}_2(u)$ labels a path, between two distinct intersection vertices of Γ.

Bennett Construction C ([2, construction 3.3] or [4, step 3]). Let

$$\Gamma_k \simeq \mathcal{S}\Gamma(X; R; s_k)$$

for some $s_k \in S_1 * S_2$, satisfying **(LBE)** but not **(RPS)**. Let $v', v'' \in \mathrm{IV}(\Gamma_k)$ be two vertices such that $\Delta_i(v') = \Delta_i(v'')$ and let $u \in U$ be such that the word $\mathrm{w}_i(u)$ labels a $v' - v''$ path. Now, by the property **(LBE)**, there exist $\tilde{v}' \in \Delta_j(v')$ and $\tilde{v}'' \in \Delta_j(v'')$ such that $\mathrm{w}_j(u)$ labels both a $v' - \tilde{v}'$ path and a $\tilde{v}'' - v''$ path. Now partition Γ_k in two subgraphs $\Gamma_k^{(1)} \supseteq \Delta_i(v')$ and $\Gamma_k^{(2)}$ so that $\mathrm{V}(\Gamma_k^{(1)}) \cap \mathrm{V}(\Gamma_k^{(2)}) = \{v''\}$, and build the X-inverse word graph Γ'_k as

$$\Gamma'_k = \left(\left(\Gamma_k^{(1)} \amalg \Gamma_k^{(2)}\right)/\xi\right)_{\mathrm{det}}, \qquad \text{with } \xi = \left\{\left((\tilde{v}', 1), (v'', 2)\right), \left((v', 1), (\tilde{v}'', 2)\right)\right\}^{\mathrm{e}}.$$

If Γ'_k is not $(X; R)$-closed, then Bennett construction A is repeated starting from Γ'_k until we obtain the $(X; R)$-closed X-inverse word graph Γ''_k. Again, if Γ''_k does not satisfy **(LBE)**, then Bennett construction B is repeated starting from it until the X-inverse word graph Γ_{k+1} satisfying **(LBE)** is obtained. ■

Let $\mathcal{A} = (\alpha, \Gamma, \beta)$ be an X-inverse automaton that satisfy **(LBE)** and **(RPS)**, let $v \in \mathrm{IV}(\Gamma)$ and $\mathrm{RP}(v) \subseteq \mathrm{V}(\Delta_1(v)) \times \mathrm{V}(\Delta_2(v))$ given by

$$\begin{aligned}\mathrm{RP}(v) = \{(v_1, v_2) \in \mathrm{V}(\Delta_1(v)) \times \mathrm{V}(\Delta_2(v)) \mid \exists u \in U \text{ such that} \\ \mathrm{w}_1(u) \in \mathrm{Lang}[(v, \Delta_1(v), v_1)],\ \mathrm{w}_2(u) \in \mathrm{Lang}[(v, \Delta_2(v), v_2)]\}.\end{aligned}$$

Given an X-inverse word graph Γ, the X-inverse word graph

$$\Gamma_{\mathrm{ass}} = \Gamma/\gamma, \qquad \text{with } \gamma = \bigcup_{v \in \mathrm{IV}(\Gamma)} \mathrm{RP}(v) = \mathrm{RP}(\mathrm{IV}(\Gamma))$$

is called the *assimilated form* of Γ. An X-inverse word graph $\Gamma \simeq \mathcal{S}\Gamma(X; R; s)$ (for some $s \in S_1 * S_2$) satisfying **(LBE)** and **(RPS)** is called *opuntoid* when it concides with its assimilated form (i.e. $\Gamma = \Gamma_{\mathrm{ass}}$) and its lobe graph LG_Γ is a tree.

Bennett Construction D ([2, section 4] or [4, step 4]). Given the X-inverse word graph $\Gamma_k \simeq \mathcal{S}\Gamma(X; R; s_k)$ for some $s_k \in S_1 * S_2$, such that Γ_k satisfies **(LBE)** and **(RPS)**, calculate its assimilated form $\Gamma_0(w) = (\Gamma_k)_{\mathrm{ass}}$. ■

Given an opuntoid X-inverse word graph Γ a *bud* is a vertex $v \in \mathrm{V}(\Gamma) \setminus \mathrm{IV}(\Gamma)$ in a lobe coloured by $i \in \{1,2\}$, such that $[\mathrm{e}_i(v)\!\uparrow_U] \neq \varnothing$. It can be easily shown that an opuntoid X-inverse word graph is $(X; R \cup W)$-closed if and only if it has no bud (see [2]).

Now, starting from the X-inverse word graph $\Gamma_0(w)$ obtained at the end of the construction D, the construction E defines a directed system $\{\Gamma_k(w)\}_{k\in\mathbb{N}}$ (in the category $\mathscr{G}_X$ of X-inverse word graphs) whose direct limit is the Schützenberger graph $\mathcal{S}\Gamma(X; R \cup W; w\eta)$.

Bennett Construction E ([2, construction 5.1] or [4, step 5]). Let $\Gamma_k(w)$ be an opuntoid X-inverse word graph that is not $(X; R \cup W)$-closed. Then a bud $v \in \mathrm{V}(\Gamma_k(w)) \setminus \mathrm{IV}(\Gamma_k(w))$ exists in a lobe Δ coloured by $i \in \{1,2\}$. Let $f = \mathrm{f}(\mathrm{e}_i(v))$, build an X-inverse word graph $\Gamma'_k(w)$ as

$$\Gamma'_k(w) = \left(\Gamma_k(w) \amalg \mathcal{S}\Gamma(X_j; R_j; f)\right)/\mu, \qquad \text{with } \mu = \{((v,1),(f,2))\}^{\mathrm{e}},$$

and calculate the assimilated form of $\Gamma'_k(w)$ as

$$\Gamma_{k+1}(w) = \Gamma'_k(w)_{\mathrm{ass}} = \Gamma'_k(w)/\,\mathrm{RP}(v'). \qquad \blacksquare$$

3. Solution of the Word Problem

We are ready to approach the word problem. We assume that the amalgam $\mathfrak{U} = [S_1, S_2; U]$ satisfies the following five conditions:

(A1) the word problem in each $S_i = \mathrm{Inv}\,\langle X_i | R_i\rangle$ is decidable,

(A2) the injective maps $\mathrm{w}_i : U \to \left(X_i \cup X_i^{-1}\right)^+$ that has been fixed are effectively calculable (clearly every finitely generated inverse semigroup U fulfills this condition),

(A3) for each $s \in S_i$ whether or not $[s\!\uparrow_U] \neq \varnothing$ is decidable, and if it is non-empty there is an effective procedure to find an element $u \in [s\!\uparrow_U]$,

(A4) the amalgam $\mathfrak{U}$ satisfies **(LB1)** and for each $e \in \mathrm{E}_{S_i}$ such that $[e\!\uparrow_U] \neq \varnothing$ the element $\mathrm{f}(e)$ is effectively calculable (recall that $\mathrm{f}(e) \in \mathrm{E}_U$ is the minimal element of $[e\!\uparrow_U]$),

(A5) the amalgam $\mathfrak{U}$ is such that Bennett construction B terminates after finitely many applications at all intersection vertices starting from an approximate X-inverse automata $\mathcal{A}$ such that $\mathcal{A} \simeq \mathcal{A}(X; R; w'\tau)$ for some $w' \in \left(X \cup X^{-1}\right)^+$ (this always happens when $\mathfrak{U}$ also satisfies the **(LB2)**, see [2, lemma 2.3]).

The word problem for $S_1 *_U S_2$ is decidable if for all $w, z \in \left(X \cup X^{-1}\right)^+$ it is decidable whether or not $z\eta \in \mathrm{Lang}[\mathcal{A}(X; R \cup W; w\eta)]$. We develop an algorithm to solve the membership problem for amalgamated free product $S_1 *_U S_2$ satisfying

(A1)–(A5) using a procedure analogous to the first three Bennett constructions, but producing each time only suitable Munn trees approximating $\mathcal{A}(X; R \cup W; w\eta)$ instead of the Schützenberger graphs of $\mathcal{R}$-classes of the free products $S_1 * S_2$. This gives the advantage of working with finite graphs.

We need to introduce some new notation. Let Γ be a finite X-inverse word graph isomorphic to a Munn Tree, and let $\Omega \subseteq \Gamma$ be an X-inverse word subgraph of Γ, which obviously must be isomorphic to a Munn Tree (recall that an X-inverse word subgraph must be strongly connected). For each pair $v_1, v_2 \in \mathrm{V}(\Omega)$, a word $\tilde{w}$ is a *spanning word* for (v_1, Ω, v_2) when it is the label of a $v_1 - v_2$ path on Ω touching each vertex in $\mathrm{V}(\Omega)$ at least once. Thus, if $\tilde{w}$ is a spanning word for (v_1, Ω, v_2), we obtain

$$(v_1, \Omega, v_2) \simeq (\varepsilon, \mathrm{MT}_X(\tilde{w}), \mathrm{r}(\tilde{w})). \tag{2}$$

Of course there are infinitely many words $\tilde{w}$ that have such a property, but it's easy to construct a calculable map

$$\varpi_\Omega : \mathrm{V}(\Omega) \times \mathrm{V}(\Omega) \to \left(X \cup X^{-1}\right)^+,$$

selecting a spanning word $\tilde{w} = \varpi_\Omega(v_1, v_2)$ for (v_1, Ω, v_2) satisfying (2).

Since

$$(\varepsilon, \mathrm{MT}_X(w), \mathrm{r}(w)) \rightsquigarrow \mathcal{A}(X; R; w\tau)$$

there exists a natural morphism

$$\psi^{(w)} : (\varepsilon, \mathrm{MT}_X(w), \mathrm{r}(w)) \to \mathcal{A}(X; R; w\tau).$$

We say that the Munn Tree $\mathrm{MT}_X(y)$ of the word y has the *good lobe ordering property* **(GLO)** if

(GLO1) $\mathrm{LG}_{\mathrm{MT}_X(y)} \simeq \mathrm{LG}_{\mathcal{S}\Gamma(X;R;y\tau)}$ by the isomorphism mapping the lobe containing the vertex $\varepsilon \in \mathrm{V}(\mathrm{MT}_X(y))$ into the lobe containing the vertex $(yy^{-1})\tau \in \mathrm{V}(\mathcal{S}\Gamma(X; R; y\tau))$,

(GLO2) for each lobe $\Delta \in \mathrm{V}(\mathrm{LG}_{\mathrm{MT}_X(y)})$ coloured by $i \in \{1, 2\}$, let $\tilde{\Delta} \supseteq \psi^{(y)}(\Delta)$ be the corresponding lobe of $\mathcal{S}\Gamma(X; R; y\tau)$, then the isomorphism

$$(v_1, \Delta, v_2) \simeq (\varepsilon, \mathrm{MT}_{X_i}(\tilde{y}), \mathrm{r}(\tilde{y}))$$

implies

$$\left(\psi^{(y)}(v_1), \tilde{\Delta}, \psi^{(y)}(v_2)\right) \simeq \mathcal{A}(X_i; R_i; \tilde{y}\tau_i),$$

for $v_1, v_2 \in \mathrm{V}(\Delta)$, $\tilde{y} = \varpi_\Delta(v_1, v_2)$.

Our first construction is analogous to Bennett construction A, but it replaces the Schützenberger graph $\mathcal{S}\Gamma(X; R; w\tau)$ by a Munn Tree $\mathrm{MT}_X(w')$, with $w'\tau = w\tau$, satisfying **(GLO)**.

Construction 1. Let $\Delta_k \in \mathrm{LG}_{\mathrm{MT}_X(w^{(k)})}$ be a lobe coloured by $i \in \{1,2\}$ of the Munn automaton $\big(\varepsilon, \mathrm{MT}_X(w^{(k)}), \mathrm{r}(w^{(k)})\big)$ approximating for $\mathcal{A}(X;R;w\tau)$. Consider two distinct intersection vertices $v_1, v_2 \in \mathrm{IV}(\mathrm{MT}_X(w^{(k)})) \cap \mathrm{V}(\Delta_k)$ such that

$$\varpi_{\Delta_k}(v_1,v_2)\tau_i = \big(\varpi_{\Delta_k}(v_1,v_2)\,\varpi_{\Delta_k}(v_1,v_2)^{-1}\big)\,\tau_i.$$

If there not exist two distinct vertices with this property, simply skip this construction. Then

$$\psi^{w^{(k)}}(v_1) = \psi^{w^{(k)}}(v_2),$$

and the Munn tree $\mathrm{MT}_X(w^{(k)})$ does not satisfy **(GLO1)**. For $j = 3-i$, the morphism $\psi^{w^{(k)}}$ maps $\Delta_j(v_1)$ and $\Delta_j(v_2)$ into the same lobe of $\mathcal{S}\Gamma(X;R;w^{(k)}\tau)$ and $\mathrm{LG}_{\mathrm{MT}_X(y)} \not\simeq \mathrm{LG}_{\mathcal{S}\Gamma(X;R;y\tau)}$ (recall that $\Delta_j(v_h)$ is the lobe adjacent to v_h that is coloured by j).

Partition $\mathrm{MT}_X(w^{(k)})$ in two subgraphs $T_k^{(1)}$ and $T_k^{(2)}$ such that

$$v_1 \in \mathrm{V}(T_k^{(1)}), \qquad \mathrm{V}(T_k^{(1)}) \cap \mathrm{V}(T_k^{(2)}) = \{v_2\}\,,$$

then build the X-inverse word graph

$$T_k = \left(\left(T_k^{(1)} \amalg T_k^{(2)}\right)/\mu\right)_{\det}, \qquad \text{with } \mu = \{((v_1,1),(v_2,2))\}^{\mathrm{e}}\,,$$

and denote by $a_k, b_k \in \mathrm{V}(T_k)$ the natural images in T_k of the vertices $\varepsilon, \mathrm{r}(w^{(k)}) \in \mathrm{V}(\mathrm{MT}_X(w^{(k)}))$. Then putting $w^{(k+1)} = \varpi_{T_k}(a_k,b_k)$ the construction terminates returning the approximate Munn automaton

$$\left(\varepsilon, \mathrm{MT}_X(w^{(k+1)}), \mathrm{r}(w^{(k+1)})\right) \simeq (a_k, T_k, b_k) \rightsquigarrow \mathcal{A}(X;R;w\tau). \qquad \blacksquare$$

Moreover each sequence of iterations of construction 1 starting from $\mathrm{MT}_X(w)$ terminates after finitely many steps returning a Munn Tree with the desired properties.

Lemma 3.1. *Let $\big(\varepsilon, \mathrm{MT}_X(w^{(k+1)}), \mathrm{r}(w^{(k+1)})\big)$ be a Munn automaton obtained by $\big(\varepsilon, \mathrm{MT}_X(w^{(k)}), \mathrm{r}(w^{(k)})\big)$ with an application of construction 1. Then $w^{(k+1)}\tau = w^{(k)}\tau$.*

Proof. For simplicity we assume that $\mathrm{MT}_X(w^{(k)})$ has only three lobes, the general case is a trivial extension of this. Let $v_1, v_2 \in \mathrm{IV}(\mathrm{MT}_X(w^{(k)}))$ such that $\Delta_i(v_1) = \Delta_i(v_2)$. We can assume without loss of generality that $\varepsilon \in \Delta_j(v_1)$ and $\mathrm{r}(w^{(k)}) \in \Delta_j(v_2)$, as shown in figure 1. Define

$$w_1^{(k)} = \varpi_{\Delta_j(v_1)}(\varepsilon, v_1), \quad w_2^{(k)} = \varpi_{\Delta_i(v_1)}(v_1,v_2), \quad w_3^{(k)} = \varpi_{\Delta_j(v_2)}(v_2, \mathrm{r}(w^{(k)}))$$

such that $v_1 = \mathrm{r}(w_1^{(k)})$ and $v_2 = \mathrm{r}(w_1^{(k)} w_2^{(k)})$. Note that $w_1^{(k)} w_2^{(k)} w_3^{(k)}$ is a spanning word for $(\varepsilon, \mathrm{MT}_X(w^{(k)}), \mathrm{r}(w^{(k)})$, so

$$\left(\varepsilon, \mathrm{MT}_X(w^{(k)}), \mathrm{r}(w^{(k)})\right) = \left(\varepsilon, \mathrm{MT}_X(w_1^{(k)} w_2^{(k)} w_3^{(k)}), \mathrm{r}(w_1^{(k)} w_2^{(k)} w_3^{(k)})\right)$$

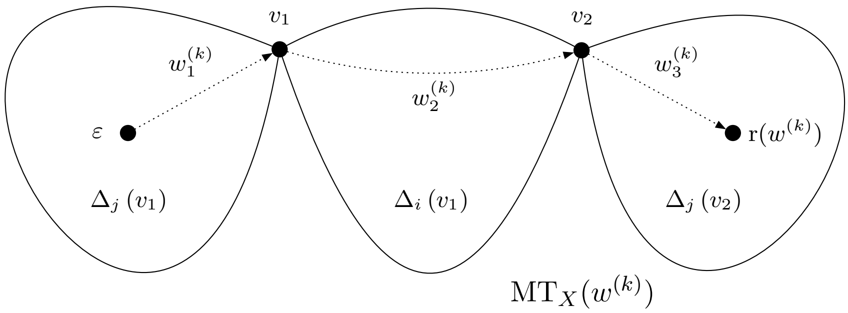

Figure 1. Situation at the beginning of construction 1.

or, equivalently,

$$\left(w_1^{(k)} w_2^{(k)} w_3^{(k)}\right) \rho = w^{(k)} \rho,$$

in particular, since $\rho \subseteq \tau$,

$$\left(w_1^{(k)} w_2^{(k)} w_3^{(k)}\right) \tau = w^{(k)} \tau.$$

The operations performed on $\mathrm{MT}_X(w^{(k)})$ to obtain $\mathrm{MT}_X(w^{(k+1)})$ give that $w^{(k+1)}$ is ρ-related (so also τ-related) to

$$w_1^{(k)} w_2^{(k)} \left(w_2^{(k)}\right)^{-1} w_3^{(k)},$$

hence

$$w^{(k+1)}\tau = \left(w_1^{(k)} w_2^{(k)} \left(w_2^{(k)}\right)^{-1} w_3^{(k)}\right) \tau = \left(w_1^{(k)}\right) \tau \left(w_2^{(k)} \left(w_2^{(k)}\right)^{-1}\right) \tau \left(w_3^{(k)}\right) \tau =$$
$$= \left(w_1^{(k)}\right) \tau \left(w_2^{(k)}\right) \tau \left(w_3^{(k)}\right) \tau = \left(w_1^{(k)} w_2^{(k)} w_3^{(k)}\right) \tau = w^{(k)} \tau. \qquad \blacksquare$$

Theorem 3.2. *Let $w^{(0)} = w$ and $\left\{w^{(k)}\right\}_{k\geq 0}$ be a sequence of words such that $\left(\varepsilon, \mathrm{MT}_X(w^{(k+1)}), \mathrm{r}(w^{(k+1)})\right)$ is obtained from $\left(\varepsilon, \mathrm{MT}_X(w^{(k)}), \mathrm{r}(w^{(k)})\right)$ with an iteration of construction 1. Then the sequence finitely terminates in a word $w^{(N)}$ such that $w^{(N)}\tau = w\tau$ and the Munn Tree $\mathrm{MT}_X(w^{(N)})$ satisfies* **(GLO)**.

Proof. Since the number of lobes of $\mathrm{MT}_X(w^{(k)})$ decreases by one for an application of construction 1, the sequence $\left\{w^{(k)}\right\}_k$ terminates in a word $w^{(N)}$, with N bounded by the number of lobes of $\mathrm{MT}_X(w^{(0)})$. From lemma 3.1 it follows that that $w^{(N)}\tau = w\tau$, that is

$$\left(\varepsilon, \mathrm{MT}_X(w^{(N)}), \mathrm{r}(w^{(N)})\right) \rightsquigarrow \mathcal{A}(X; R; w\tau), \tag{3}$$

Given an arbitrary lobe $\Delta \subseteq \mathrm{MT}_X(w^{(N)})$ coloured by $i \in \{1,2\}$ and two intersection vertices $v_1, v_2 \in \mathrm{IV}(\mathrm{MT}_X(w^{(N)})) \cap \mathrm{V}(\Delta)$, it follows that

$$(v_1, \Delta, v_2) \rightsquigarrow \mathcal{A}(X_i; R_i; \varpi_\Delta(v_1, v_2)\tau_i) \simeq \mathcal{A}(X; R; \varpi_\Delta(v_1, v_2)\tau).$$

From the properties of $\mathrm{MT}_X(w^{(N)})$ we know that

$$\varpi_\Delta(v_1, v_2)\tau_i \neq \left(\varpi_\Delta(v_1, v_2)\,\varpi_\Delta(v_1, v_2)^{-1}\right)\tau_i,$$

hence

$$\varpi_\Delta(v_1, v_2)\tau \neq \left(\varpi_\Delta(v_1, v_2)\,\varpi_\Delta(v_1, v_2)^{-1}\right)\tau. \tag{4}$$

so $\varpi_\Delta(v_1, v_2)\tau \notin \mathrm{E}_{S_1 * S_2}$. Now consider an arbitrary approximate X-inverse automaton

$$(\alpha, \Gamma, \beta) \rightsquigarrow \mathcal{A}(X; R; \varpi_\Delta(v_1, v_2)\tau).$$

The $(X;R)$-closure (see Stephen [13]) of this automaton is the Schützenberger automaton of $\varpi_\Delta(v_1, v_2)\tau$, and the natural images of $\alpha, \beta \in \mathrm{V}(\Gamma)$ are respectively

$$\left(\varpi_\Delta(v_1, v_2)\,\varpi_\Delta(v_1, v_2)^{-1}\right)\tau, \varpi_\Delta(v_1, v_2)\tau \in \mathrm{V}(\mathcal{S}\Gamma(X; R; \varpi_\Delta(v_1, v_2)\tau)).$$

Condition (4) guarantees that $\alpha \neq \beta$, hence no sequence of $(X;R)$-expansions applied from Δ can identify v_1 and v_2. Since the lobe graph $\mathrm{LG}_{\mathrm{MT}_X(w^{(N)})}$ is a tree, no sequence of $(X;R)$-expansions applied starting from $\mathrm{MT}_X(w^{(N)})$ can identify v_1 and v_2. This means that each $(X;R)$-expansion on $\mathrm{MT}_X(w^{(N)})$ operates only inside the lobes, leaving unchanged the lobe graph. This implies **(GLO1)**. Since $(X;R)$-expansions operate only inside the lobes, equation (4) gives condition **(GLO2)**. ■

Remark 3.1. The Munn tree of the word $w' = w^{(N)}$ *does not represent a normal form* for the τ-class of w. In fact, let $v \in \left(X \cup X^{-1}\right)^+$ such that $v \neq w$ and $v\tau = w\tau$, iterations of construction 1 starting from $\mathrm{MT}_X(v)$ may terminate in $\mathrm{MT}_X(v^{(N')})$ satisfying **(GLO)** with $\mathrm{MT}_X(v^{(N')}) \neq \mathrm{MT}_X(w^{(N)})$. ■

The next construction closely follows Bennett construction B, but it works, again, on Munn trees, hence the finiteness property of the produced graphs is guaranteed. The correspondence between the Munn automaton of $w^{(k)}$ and the Schützenberger automaton of $w^{(k)}\tau$ is given by the morphism (in the category $\mathscr{A}_X$ of X-inverse automata)

$$\psi^{(w^{(k)})} : \left(\varepsilon, \mathrm{MT}_X(w^{(k)}), \mathrm{r}(w^{(k)})\right) \to \mathcal{A}(X; R; w^{(k)}\tau).$$

Construction 2 (Expansion of the intersections). Given the approximate Munn automaton

$$\left(\varepsilon, \mathrm{MT}_X(w^{(k)}), \mathrm{r}(w^{(k)})\right) \rightsquigarrow \mathcal{A}(X; R; w\tau),$$

let $v \in \mathrm{IV}(\mathrm{MT}_X(w^{(k)}))$ be such that

$$[\varpi_{\Delta_i(v)}(v,v)\tau_i\big\uparrow_U] \neq \varnothing,$$

for an $i \in \{1,2\}$, and

$$[\varpi_{\Delta_j(v)}(v,v)\tau_j\Big\uparrow_U] = \varnothing \quad \text{or} \quad \mathrm{f}\left(\varpi_{\Delta_i(v)}(v,v)\tau_i\right) \neq \mathrm{f}\left(\varpi_{\Delta_j(v)}(v,v)\tau_j\right).$$

with $j = 3-i$. Let $f = \mathrm{f}\left(\varpi_{\Delta_i(v)}(v,v)\tau_i\right)$. Build the X-inverse word graph T_k as

$$T_k = \left(\left(\mathrm{MT}_X(w^{(k)}) \amalg \mathrm{MT}_{X_j}(\mathrm{w}_j(f))\right)/\zeta\right)_{\det}, \quad \text{with } \zeta = \{((v,1),(\varepsilon,2))\}^{\mathrm{e}},$$

and as usual denote by $a_k, b_k \in \mathrm{V}(T_k)$ the respective images of $\varepsilon, \mathrm{r}(w^{(k)}) \in \mathrm{V}(\mathrm{MT}_X(w^{(k)}))$. Then

$$\left(\varepsilon, \mathrm{MT}_X(\tilde{w}^{(k)}), \mathrm{r}(\tilde{w}^{(k)})\right) \simeq (a_k, T_k, b_k),$$

with $\tilde{w}^{(k)} = \varpi_{T_k}(a_k, b_k)$. If the Munn tree $\mathrm{MT}_X(\tilde{w}^{(k)})$ does not satisfy **(GLO)**, apply all possible iterations of construction 1 to obtain $\mathrm{MT}_X(w^{(k+1)})$, getting the Munn automaton

$$\left(\varepsilon, \mathrm{MT}_X(w^{(k+1)}), \mathrm{r}(w^{(k+1)})\right).$$

■

Before stating the main result, we need a technical lemma.

Lemma 3.3. *Let $\mathrm{MT}_X(w^{(k)})$ be a Munn tree that satisfies* **(GLO)**, *and let $\mathcal{A}_k$ be the Schützenberger automaton*

$$\mathcal{A}_k = (\alpha_k, \Gamma_k, \beta_k) = \mathcal{A}(X; R; w^{(k)}\tau).$$

Let $\mathrm{MT}_X(w^{(k+1)})$ be obtained from $\mathrm{MT}_X(w^{(k)})$ applying the construction 2 to the vertex $v \in \mathrm{IV}(\mathrm{MT}_X(w^{(k)}))$. Let $\mathcal{A}_{k+1} = (\alpha_{k+1}, \Gamma_{k+1}, \beta_{k+1})$ the X-inverse automaton obtained applying Bennett construction B to the vertex

$$v' = \psi^{(w^{(k)})}(v) \in \mathrm{IV}(\Gamma_k).$$

Then

$$\mathcal{A}_{k+1} = (\alpha_{k+1}, \Gamma_{k+1}, \beta_{k+1}) \simeq \mathcal{A}(X; R; w^{(k+1)}\tau).$$

Proof. Let $h = \varpi_{\mathrm{MT}_X(w^{(k)})}(v, \varepsilon)$. The Munn automaton $(\varepsilon, \mathrm{MT}_X(w^{(k)}), \mathrm{r}(w^{(k)}))$ approximates $\mathcal{A}_k$, i.e. $(\varepsilon, \mathrm{MT}_X(w^{(k)}), \mathrm{r}(w^{(k)})) \rightsquigarrow \mathcal{A}_k$, thus we obtain

$$\left(\varepsilon, \mathrm{MT}_X(hw^{(k)}), \mathrm{r}(hw^{(k)})\right) \simeq \left(v, \mathrm{MT}_X(w^{(k)}), \mathrm{r}(w^{(k)})\right) \rightsquigarrow$$
$$\rightsquigarrow \left(v', \Gamma_k, \beta_k\right) \simeq \mathcal{A}(X; R; (hw^{(k)})\tau).$$

With the same notation as in Bennett construction B and construction 2 we have $f = \mathrm{f}\left(\varpi_{\Delta_i(v)}(v,v)\tau_i\right) = \mathrm{f}(\mathrm{e}_i(v'))$. The X-inverse word graphs Γ'_k and T_k can be similarly built as

$$\begin{aligned}\left(\alpha'_k,\Gamma'_k,\beta'_k\right) &\simeq \mathcal{A}(X;R;f) \bullet_{\det} \left(v',\Gamma_k,\beta_k\right) \rightsquigarrow \mathcal{A}(X;R;f\,(hw^{(k)})\tau),\\ \left(a'_k,T_k,b_k\right) &\simeq \left(\varepsilon,\mathrm{MT}_{X_j}(\mathrm{w}_j(f)\,\mathrm{w}_j(f)^{-1}),\varepsilon\right) \bullet_{\det} \left(v,\mathrm{MT}_X(w^{(k)}),\mathrm{r}(w^{(k)})\right) \simeq\\ &\simeq \left(\varepsilon,\mathrm{MT}_X(\mathrm{w}_j(f)\,\mathrm{w}_j(f)^{-1}hw^{(k)}),\mathrm{r}(hw^{(k)})\right).\end{aligned}$$

Here the symbol $\bullet_{\det}$ denotes the determinized form of the product and it is easily defined as follows: given two X-inverse automata (α,Γ,β) and (γ,Ω,δ), we define $(\alpha,\Gamma,\beta)\bullet_{\det}(\gamma,\Omega,\delta) = (\tilde{\alpha},\Lambda,\tilde{\delta})$, where

$$\Lambda = ((\Gamma \amalg \Omega)/\mu)_{\det}\,, \qquad \text{with } \mu = \{((\beta,1),(\gamma,2))\}^{\mathrm{e}}\,,$$

and $\tilde{\alpha},\tilde{\beta}$ are the natural projection of α,β.

The X-inverse word graph Γ_{k+1} obtained in Bennett construction A is the underlying graph of the $(X;R)$-closure of $(\alpha'_k,\Gamma'_k,\beta'_k)$, thus

$$\left(\alpha'_k,\Gamma'_k,\beta'_k\right) \rightsquigarrow (\tilde{\alpha}_{k+1},\Gamma_{k+1},\beta_{k+1}) \simeq \mathcal{A}(X;R;f\,(hw^{(k)})\tau).$$

Then the word h labels a $\tilde{\alpha}_{k+1}-\alpha_{k+1}$ path on Γ_{k+1} for some $\alpha_{k+1}\in \mathrm{V}(\Gamma_{k+1})$, and it is easy to verify that α_{k+1} is the natural image of $\alpha_k \in \mathrm{V}(\Gamma_k)$, thus

$$(\alpha_{k+1},\Gamma_{k+1},\beta_{k+1}) \simeq \mathcal{A}(X;R;h^{-1}\tau f\,(hw^{(k)})\tau).$$

Since f is an idempotent, we observe that

$$\left(\mathrm{w}_j(f)\,\mathrm{w}_j(f)^{-1}hw^{(k)}\right)\tau = ff^{-1}(hw^{(k)})\tau = f\,(hw^{(k)})\tau, \tag{5}$$

thus $(a'_k,T_k,b_k) \rightsquigarrow \mathcal{A}(X;R;f\,(hw^{(k)})\tau)$. From the definition of T_k we know that the word h labels an $a'_k - a_k$ path on T_k for some $a_k \in \mathrm{V}(T_k)$. The vertex a_k is clearly the natural image of $\varepsilon \in \mathrm{V}(\mathrm{MT}_X(w^{(k)}))$, and it results

$$(a_k,T_k,b_k) \simeq \left(\varepsilon,\mathrm{MT}_X(\tilde{w}^{(k)}),\mathrm{r}(\tilde{w}^{(k)})\right), \qquad \text{with } \tilde{w}^{(k)} = h^{-1}\,\mathrm{w}_j(f)\,\mathrm{w}_j(f)^{-1}hw^{(k)}.$$

From equation (5) and for the well-known properties of the approximate automata we obtain

$$(a_k,T_k,b_k) \rightsquigarrow \mathcal{A}(X;R;h^{-1}\tau f\,(hw^{(k)})\tau).$$

Construction 2 terminates applying a sequence of iterations of construction 1 on T_k, returning the Munn tree $\mathrm{MT}_X(w^{(k+1)})$ which satisfies **(GLO)** and

$$w^{(k+1)}\tau = \tilde{w}^{(k)}\tau = h^{-1}\tau\,f\,(hw^{(k)})\tau.$$

Then the claim follows from

$$(\alpha_{k+1},\Gamma_{k+1},\beta_{k+1}) \simeq \mathcal{A}(X;R;h^{-1}\tau\,f\,(hw^{(k)})\tau) = \mathcal{A}(X;R;w^{(k+1)}\tau). \qquad \blacksquare$$

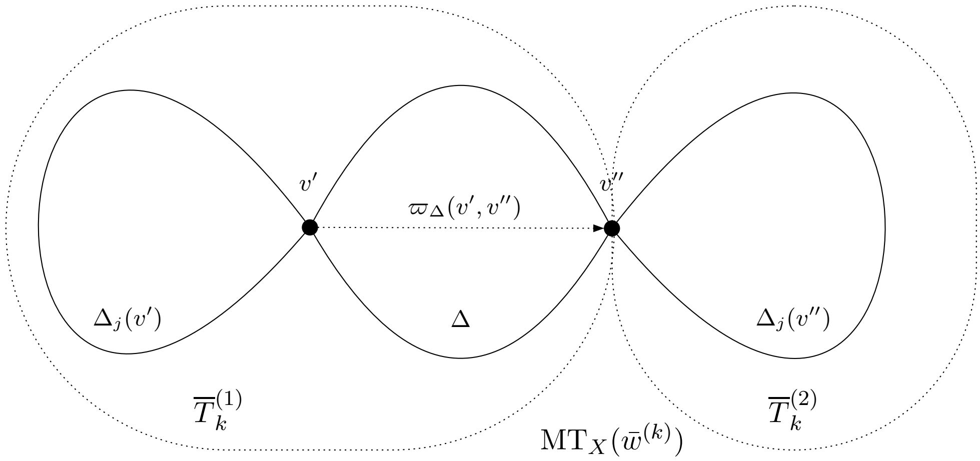

Figure 2. Situation at the beginning of construction 3.

From the previous lemma and condition **(A5)** we can easily derive the main result.

Theorem 3.4. *Let $\left(\varepsilon, \mathrm{MT}_X(w^{(N)}), \mathrm{r}(w^{(N)})\right)$ be the Munn automaton with* **(GLO)** *obtained at the end of construction 1, and let $\left\{w^{(k)}\right\}_{k\geq N}$ be a sequence of words such that the Munn automaton of $w^{(k+1)}$ is obtained from the Munn automaton of $w^{(k)}$ with one application of construction 2. Then the sequence finitely terminates in a word $w^{(M)} = \overline{w}$, with $M \geq N$, such that*

$$(\varepsilon, \mathrm{MT}_X(\overline{w}), \mathrm{r}(\overline{w})) \rightsquigarrow \mathcal{A}(X; R; \overline{w}\tau) \rightsquigarrow \mathcal{A}(X; R \cup W; \overline{w}\eta) = \mathcal{A}(X; R \cup W; w\eta),$$

and the Schützenberger automaton $\mathcal{A}(X; R; \overline{w}\tau)$ satisfies **(LBE)**. ∎

Now we describe the last construction (an analogue of Bennett construction C). Our task is to obtain a Munn automaton approximating the Schützenberger automaton of an element $\tilde{w}\tau \in S_1 * S_2$ and satisfying **(RPS)** such that $\tilde{w}\eta = w\eta$.

Construction 3 (Lobe Separation). Let

$$\left(\varepsilon, \mathrm{MT}_X(\bar{w}^{(k)}), \mathrm{r}(\bar{w}^{(k)})\right) \rightsquigarrow \mathcal{A}(X; R; \bar{w}^{(k)}\tau)$$

such that $\mathcal{A}(X; R; \bar{w}^{(k)}\tau)$ satisfies **(LBE)** and $\mathrm{MT}_X(\bar{w}^{(k)})$ satisfies **(GLO)**. Let $\Delta \subseteq \mathrm{MT}_X(\bar{w}^{(k)})$ be a lobe coloured by $i \in \{1, 2\}$ such that

$$[\varpi_\Delta(v', v'')\tau_i{\upharpoonleft}_U] \neq \varnothing, \tag{6}$$

for distinct $v', v'' \in \mathrm{IV}(\mathrm{MT}_X(\bar{w}^{(k)})) \cap \mathrm{V}(\Delta)$ (see figure 2). For

$$u \in [\varpi_\Delta(v', v'')\tau_i{\upharpoonleft}_U]$$

the word $\mathrm{w}_i(u)$ labels a $\psi^{(\bar{w}^{(k)})}(v') - \psi^{(\bar{w}^{(k)})}(v'')$ path on the Schützenberger automaton $\mathcal{A}(X;R;\bar{w}^{(k)}\tau)$.

Partition $\mathrm{MT}_X(\bar{w}^{(k)})$ in two subgraphs $\overline{T}_k^{(1)}$ and $\overline{T}_k^{(2)}$ such that

$$v' \in \mathrm{V}(\overline{T}_k^{(1)}), \qquad \mathrm{V}(\overline{T}_k^{(1)}) \cap \mathrm{V}(\overline{T}_k^{(2)}) = \{v''\},$$

then build the X-inverse word graph

$$\overline{T}_k = \left(\left(\overline{T}_k^{(1)} \amalg \overline{T}_k^{(2)} \amalg \mathrm{MT}_{X_j}(\mathrm{w}_j(u))\right)/\xi\right)_{\det},$$

where

$$\xi = \left\{\left((v'',2),(\mathrm{r}(\mathrm{w}_j(u)),3)\right), \left((v',1),(\varepsilon,3)\right)\right\}^{\mathrm{e}}.$$

As usual denote by $\bar{a}_k, \bar{b}_k \in \mathrm{V}(\overline{T}_k)$ the natural images of $\varepsilon, \mathrm{r}(\bar{w}^{(k)}) \in \mathrm{V}(\mathrm{MT}_X(\bar{w}^{(k)}))$.

The X-inverse word graph $\overline{T}_k$ is isomorphic to the Munn tree of $\breve{w}^{(k+1)} = \varpi_{\overline{T}_k}(\bar{a}_k, \bar{b}_k)$. If the Munn tree $\mathrm{MT}_X(\breve{w}^{(k+1)})$ does not satisfies **(GLO)** and if its $(X;R)$-closure $\mathcal{S}\Gamma(X;R;\breve{w}^{(k+1)}\tau)$ does not satisfies **(LBE)** iteratively apply constructions 1 and 2 (starting from $\mathrm{MT}_X(\breve{w}^{(k+1)})$) obtaining as a result the Munn automaton

$$\left(\varepsilon, \mathrm{MT}_X(\bar{w}^{(k+1)}), \mathrm{r}(\bar{w}^{(k+1)})\right). \qquad \blacksquare$$

The next result is analogous to lemma 3.3, so we omit the proof (close to the proof of the previous lemma).

Theorem 3.5. *Let* $\mathrm{MT}_X(\bar{w}^{(k)})$ *be a Munn tree and*

$$\overline{\mathcal{A}}_k = \left(\overline{\alpha}_k, \Gamma_k, \overline{\beta}_k\right) = \mathcal{A}(X;R;\bar{w}^{(k)}\tau)$$

such that $\mathrm{MT}_X(\bar{w}^{(k)})$ *satisfies* **(GLO)** *and* $\overline{\mathcal{A}}_k$ *satisfies* **(LBE)**. *Let*

$$v', v'' \in \mathrm{IV}(\mathrm{MT}_X(\bar{w}^{(k)})) \cap \mathrm{V}(\Delta)$$

for some lobe $\Delta \in \mathrm{LG}_{\mathrm{MT}_X(\bar{w}^{(k)}))}$ *colored by* i*, such that*

$$[\varpi_\Delta(v', v'')\tau_i \!\upharpoonright_U] \neq \varnothing.$$

Let $\mathrm{MT}_X(\bar{w}^{(k+1)})$ *be the Munn tree obtained from* $\mathrm{MT}_X(\bar{w}^{(k)})$ *by an application of construction 3 to* v', v''*. Let*

$$\bar{v}' = \psi^{(\bar{w}^{(k)})}(v') \in \mathrm{IV}(\Gamma_k), \qquad \bar{v}'' = \psi^{(\bar{w}^{(k)})}(v'') \in \mathrm{IV}(\Gamma_k),$$

and let $\overline{\mathcal{A}}_{k+1} = \left(\overline{\alpha}_{k+1}, \Gamma_{k+1}, \overline{\beta}_{k+1}\right)$ *be the* X*-inverse automaton obtained from* $\overline{\mathcal{A}}_k$ *with one application of Bennett construction C to vertices* $\bar{v}', \bar{v}''$*. Then*

$$\overline{\mathcal{A}}_{k+1} = \left(\overline{\alpha}_{k+1}, \Gamma_{k+1}, \overline{\beta}_{k+1}\right) \simeq \mathcal{A}(X;R;\bar{w}^{(k+1)}\tau). \qquad \blacksquare$$

Consider the word $w \in \left(X \cup X^{-1}\right)^+$. The iterations of the first three constructions starting from $\mathrm{MT}_X(w)$ lead (in a finite number of steps) to the Munn Tree $\mathrm{MT}_X(\widehat{w})$, such that $w\eta = \widehat{w}\eta$, $\mathrm{MT}_X(\widehat{w})$ satisfies **(GLO)** and $\mathcal{A}(X; R; \widehat{w}\tau)$ satisfies **(LBE)**, **(RPS)**. If we apply (the iterations of) the first three Bennett constructions starting from $(\varepsilon, \mathrm{MT}_X(w), \mathrm{r}(w))$ we would obtain an X-inverse automaton (cactoid, that satisfies **(LBE)** and **(RPS)**) isomorphic to $\mathcal{A}(X; R; \widehat{w}\tau)$, and with the fourth Bennett construction we obtain its assimilated form

$$\mathcal{A}_0(w) \simeq \mathcal{A}(X; R; \widehat{w}\tau)_{\mathrm{ass}} \rightsquigarrow \mathcal{A}(X; R \cup W; w\eta).$$

At this point consider the second word $z \in \left(X \cup X^{-1}\right)^+$ which factorizes as

$$z = z_{(1)} z_{(2)} \dots z_{(N)},$$

where the factors $z_{(k)}$ are alternatively in $\left(X_1 \cup X_1^{-1}\right)^+$ and $\left(X_2 \cup X_2^{-1}\right)^+$ for each $k \in \{1, ..., n\}$. Now we will demonstrate how to "simulate" the reading of z on the Schützenberger automaton $\mathcal{A}(X; R \cup W; w\eta)$ using an appropriate sequence of Munn Trees $\{\mathrm{MT}_X(\widehat{w}_k)\}_k$.

Starting with $k = 1$, we define

$$\widehat{w}_1 = \widehat{w}, \qquad v_1 = \varepsilon \in \mathrm{V}(\mathrm{MT}_X(\widehat{w})),$$

then we explain how to carry out the k^{th} iteration.

Let $i \in \{1, 2\}$ such that $z_{(k)} \in \left(X_i \cup X_i^{-1}\right)^+$, and let $j = 3 - i$. We denote by $\Delta_k \subseteq \mathrm{MT}_X(\widehat{w}_k)$ the lobe containing the vertex $v_k \in \mathrm{V}(\mathrm{MT}_X(\widehat{w}_k))$, i.e. $v_k \in \mathrm{V}(\Delta_k)$. If v_k is an intersection vertex we call Δ_k the adjacent lobe coloured like $z_{(k)}$, i.e. $\Delta_k = \Delta_i(v_k)$. We define

$$\tilde{\Delta}_k \supseteq \psi^{(\widehat{w}_k)}(\Delta_k), \quad \tilde{v}_k = \psi^{(\widehat{w}_k)}(v_k).$$

Then we have the following mutually exclusive cases:

(1) Δ_k is coloured by i,

(2) Δ_k is coloured by j.

In case (1) we need to verify if the factor $z_{(k)}$ labels a path in lobe $\tilde{\Delta}_k$ starting from vertex $\tilde{v}_k$. For this purpose we make use of the following result.

Lemma 3.6. *The factor $z_{(k)} \in \left(X_i \cup X_i^{-1}\right)^+$ labels a path in lobe $\tilde{\Delta}_k \subseteq \mathcal{S}\Gamma(X; R; \widehat{w}_k\tau)$ starting from vertex $\tilde{v}_k \in \mathrm{V}(\mathcal{S}\Gamma(X; R; \widehat{w}_k\tau))$ if and only if*

$$\left(z_{(k)} z_{(k)}^{-1} \varpi_{\Delta_k}(v_k, v_k)\right) \tau_i = \varpi_{\Delta_k}(v_k, v_k) \tau_i. \tag{7}$$

Proof. Since Δ_k is isomorphic to a Munn tree, it follows that

$$(v_k, \Delta_k, v_k) \simeq (\varepsilon, \mathrm{MT}_X(\varpi_{\Delta_k}(v_k, v_k)), \mathrm{r}(\varpi_{\Delta_k}(v_k, v_k))) \rightsquigarrow \mathcal{A}(X_i; R_i; \varpi_{\Delta_k}(v_k, v_k)\tau_i).$$

From the properties of $\widehat{w}_k$ each sequence of $(X;R)$-expansions applied starting from $\mathrm{MT}_X(\widehat{w}_k)$ works only internally to the lobes, so the $(X;R)$-closure of $\mathrm{MT}_X(\widehat{w}_k)$ transforms the lobe Δ_k in $\tilde{\Delta}_k$, whence

$$\mathcal{A}(X_i; R_i; \varpi_{\Delta_k}(v_k, v_k)\tau_i) \simeq \left(\tilde{v}_k, \tilde{\Delta}_k, \tilde{v}_k\right).$$

The claim then follows from the well-known properties of the Schützenberger automata. ■

We remark that it's possible to verify condition (7) thanks to **(A1)**. If we obtain

$$\left(z_{(k)} z_{(k)}^{-1} \varpi_{\Delta_k}(v_k, v_k)\right) \tau_i \neq \varpi_{\Delta_k}(v_k, v_k)\tau_i,$$

then $z\eta \not\geq w\eta$ and we can terminate the entire procedure answering $z\eta \neq w\eta$. Otherwise if

$$\left(z_{(k)} z_{(k)}^{-1} \varpi_{\Delta_k}(v_k, v_k)\right) \tau_i = \varpi_{\Delta_k}(v_k, v_k)\tau_i,$$

we "simulate" the reading of $z_{(k)}$ on $\mathcal{A}(X; R; \widehat{w}_k\tau)$ starting from the vertex $\tilde{v}_k$ building a new word $\widehat{w}_{k+1} \in \left(X \cup X^{-1}\right)^+$ such that

- $\widehat{w}_{k+1}\eta = \widehat{w}_k\eta$,
- $\mathrm{V}(\mathrm{MT}_X(\widehat{w}_{k+1})) \supseteq \mathrm{V}(\mathrm{MT}_X(\widehat{w}_k))$,
- $\mathrm{MT}_X(\widehat{w}_{k+1})$ contains a $v_k - \mathrm{r}(v_k z_{(k)})$ path labelled with $z_{(k)}$.

We build $\widehat{w}_{k+1}$ as

$$\widehat{w}_{k+1} = v_k z_{(k)} z_{(k)}^{-1} v_k^{-1} \widehat{w}_k, \qquad v_{k+1} = \mathrm{r}(v_k z_{(k)}),$$

then we proceed with the $(k+1)^{\mathrm{th}}$ iteration.

Now we consider case (2) in which Δ_k is coloured by $j = 3 - i$. The factor $z_{(k)}$ labels a path in a $(X; R \cup W)$-expanded form[4] of $\mathcal{A}(X; R; \widehat{w}_k\tau)$ starting from the (image of) vertex $\tilde{v}_k$ if and only if $(\tilde{v}_k, \tilde{v}''_k) \in \mathrm{RP}(\tilde{v}'_k)$ for some pair of vertices $\tilde{v}'_k, \tilde{v}''_k \in \mathrm{V}(\mathcal{S}\Gamma(X; R; \widehat{w}_k\tau)) \setminus \{\tilde{v}_k\}$, or $\tilde{v}_k$ is a bud. So it's clear that if v_k is an intersection vertex, i.e. $v_k \in \mathrm{IV}(\mathrm{MT}_X(\widehat{w}_k))$, then the two possibilities fail to be satisfied, and the entire procedure terminates answering $z\eta \not\geq w\eta$, in particular $z\eta \neq w\eta$. So we assume that v_k is *not* an intersection vertex, i.e.

$$v_k \in \mathrm{V}(\mathrm{MT}_X(\widehat{w}_k)) \setminus \mathrm{IV}(\mathrm{MT}_X(\widehat{w}_k)).$$

Again, we have three mutually exclusive cases:

[4]We say that an X-inverse word graph Γ' is an $(X;R)$-*expanded form* of the X-inverse word graph Γ if it is obtained from this with a finite sequence of $(X;R)$-expansions.

(2a) $[\varpi_{\Delta_k}(v_k, v'_k)\tau_j\!\upharpoonleft_U] = \varnothing$ for each $v'_k \in \mathrm{V}(\Delta_k)$,

(2b) $[\varpi_{\Delta_k}(v_k, v'_k)\tau_j\!\upharpoonleft_U] \neq \varnothing$ for a (unique) intersection vertex $v'_k \in \mathrm{IV}(\Delta_k)$,

(2c) a vertex $v'_k \in \mathrm{V}(\Delta_k)$ exists such that $[\varpi_{\Delta_k}(v_k, v'_k)\tau_j\!\upharpoonleft_U] \neq \varnothing$, but there is no vertex $v''_k \in \mathrm{IV}(\Delta_k)$ such that $[\varpi_{\Delta_k}(v_k, v''_k)\tau_j\!\upharpoonleft_U] \neq \varnothing$.

In case (2a) the vertex v_k is not a bud and does not occur in any element of the relation $\mathrm{RP}(\tilde{v}'_k)$ for any vertex $\tilde{v}'_k \in \mathrm{IV}(\mathcal{A}(X; R; \widehat{w}_k\tau))$. Then no sequence of $(X; R \cup W)$-expansions could introduce a path labelled by $z_{(k)}$ starting from (an image of) v_k, whence we can terminate the entire procedure answering $z\eta \ngeq w\eta$, in particular $z\eta \neq w\eta$.

In case (2b), the image of the vertex $\tilde{v}_k$ in the assimilated form $\mathcal{A}(X; R; \widehat{w}_k\tau)_{\mathrm{ass}}$ is an intersection vertex $\tilde{\tilde{v}}_k \in \mathrm{IV}(\mathcal{A}(X; R; \widehat{w}_k\tau)_{\mathrm{ass}})$.

Putting $\tilde{v}'_k = \psi^{(\widehat{w}_k)}(v'_k) \in \mathrm{V}(\mathcal{S}\Gamma(X; R; \widehat{w}_k\tau))$, a vertex $\tilde{v}''_k \in \mathrm{V}(\Delta_i(\tilde{v}'_k))$ exists such that $(\tilde{v}_k, \tilde{v}''_k) \in \mathrm{RP}(\tilde{v}'_k)$. We build the word $\widehat{w}'_k \in \left(X \cup X^{-1}\right)^+$ such that

- $\widehat{w}'_k\eta = \widehat{w}_k\eta$,
- $\mathrm{V}(\mathrm{MT}_X(\widehat{w}'_k)) \supseteq \mathrm{V}(\mathrm{MT}_X(\widehat{w}_k))$,
- $\mathrm{MT}_X(\widehat{w}'_k)$ contains a vertex v''_k whose image in $\mathcal{A}(X; R; \widehat{w}_k\tau)$ is $\tilde{v}''_k$, i.e. $\tilde{v}''_k = \psi^{(\widehat{w}_k)}(v''_k)$.

For $u \in [\varpi_{\Delta_k}(v_k, v'_k)\tau_j\!\upharpoonleft_U]$, the words $\mathrm{w}_j(u)$ and $\mathrm{w}_i(u)^{-1}$ label respectively a $\tilde{v}_k - \tilde{v}'_k$ path and a $\tilde{v}'_k - \tilde{v}''_k$ path on $\mathcal{A}(X; R; \widehat{w}_k\tau)$. We proceed putting

$$\widehat{w}'_k = v'_k \,\mathrm{w}_i(u)^{-1}\,\mathrm{w}_i(u)(v'_k)^{-1}\widehat{w}_k, \qquad v''_k = \mathrm{r}(v'_k \,\mathrm{w}_i(u)^{-1}),$$

and we call $\Delta'_k \subseteq \mathrm{MT}_X(\widehat{w}'_k)$ the lobe of $\mathrm{MT}_X(\widehat{w}'_k)$ containing the vertex v'_k. We remark that $v'_k \in \mathrm{V}(\Delta_i(v'_k)) \subseteq \mathrm{V}(\Delta'_k)$. Now we can terminate the current iteration applying case (1) with $\widehat{w}'_k, v''_k, z_{(k)}$: if we have

$$\left(z_{(k)}z_{(k)}^{-1}\,\varpi_{\Delta'_k}(v''_k, v''_k)\right)\tau_i \neq \varpi_{\Delta'_k}(v''_k, v''_k)\tau_i$$

we terminate the entire procedure answering $z\eta \ngeq w\eta$, hence $z\eta \neq w\eta$; otherwise if we have

$$\left(z_{(k)}z_{(k)}^{-1}\,\varpi_{\Delta'_k}(v''_k, v''_k)\right)\tau_i = \varpi_{\Delta'_k}(v''_k, v''_k)\tau_i$$

we proceed with the $(k+1)^{\mathrm{th}}$ iteration with

$$\widehat{w}_{k+1} = v''_k z_{(k)} z_{(k)}^{-1}(v''_k)^{-1}\widehat{w}'_k, \qquad v_{k+1} = \mathrm{r}(v''_k z_{(k)}).$$

In the last case (2c), we remark that, for an arbitrary $u \in [\varpi_{\Delta_k}(v_k, v'_k)\tau_j\!\upharpoonleft_U]$, $uu^{-1} \in [\varpi_{\Delta_k}(v_k, v_k)\tau_j\!\upharpoonleft_U] \neq \varnothing$, and since v'_k cannot be an intersection vertex then it is a bud. So we take

$$f = \mathrm{f}\left(\varpi_{\Delta_k}(v_k, v_k)\tau_j\right) \in \mathrm{E}_U,$$

and we verify whether or not $z_{(k)}$ labels or not a path on $\mathcal{A}(X_i, R_i; f)$ starting from the vertex $f \in \mathrm{V}(\mathcal{S}\Gamma(X_i, R_i; f))$. To perform this verification we make use of lemma 3.6: since f is an idempotent we have $\left(\mathrm{w}_i(f)\,\mathrm{w}_i(f)^{-1}\right)\tau_i = ff^{-1} = f = \mathrm{w}_i(f)\tau_i$, so if

$$\left(z_{(k)} z_{(k)}^{-1}\,\mathrm{w}_i(f)\right)\tau_i \neq \mathrm{w}_i(f)\tau_i$$

we terminate the entire procedure answering $z\eta \not\geq w\eta$, in particular $z\eta \neq w\eta$. If

$$\left(z_{(k)} z_{(k)}^{-1}\,\mathrm{w}_i(f)\right)\tau_i = \mathrm{w}_i(f)\tau_i$$

we "simulate" the reading of $z_{(k)}$ on the new lobe $\mathcal{S}\Gamma(X_i; R_i; f)$ glued to $\mathcal{A}(X; R; \widehat{w}_k\tau)$ in the vertex $\tilde{v}_k$: we assign

$$\widehat{w}_{k+1} = v_k\,\mathrm{w}_i(f)\,\mathrm{w}_i(f)^{-1} z_{(k)} z_{(k)}^{-1} v_k^{-1}\widehat{w}_k, \qquad v_{k+1} = \mathrm{r}(v_k z_{(k)}),$$

and we pass to the $(k+1)^{\text{th}}$ iteration.

Now suppose that we perform[5] N iterations without establishing that $z\eta \neq w\eta$. If we have $v_N \neq \mathrm{r}(\widehat{w}_N)$ we terminate answering $z\eta \not\geq w\eta$, in particular $z\eta \neq w\eta$. Otherwise if we have $v_N = \mathrm{r}(\widehat{w}_N)$, the word z labels a $(ww^{-1})\eta - w\eta$ path on the Schützenberger automaton $\mathcal{A}(X; R \cup W; w\eta)$, hence we obtain $z\eta \geq w\eta$. In this latest case we have to repeat the entire procedure inverting the roles of z and w: if we obtain $z\eta \geq w\eta$, then we can assert $z\eta = w\eta$.

4. A concrete realization

We conclude the paper providing a very simple example of inverse semigroup amalgam $\mathfrak{U} = [S_1, S_2; U]$, where $S_i = \mathrm{Inv}\,\langle X_i | R_i\rangle$, such that it satisfies **(A1)**,...,**(A5)** and each S_i has infinite $\mathcal{R}$-classes (that is infinite Schützenberger graphs). Let

$$X_i = \{a_i, b_i, c_i\}, \qquad R_i = \left\{(a_i a_i^{-1}, a_i a_i a_i^{-1} b_i b_i^{-1} a_i^{-1})\right\}$$

so that the semigroups $S_i = \mathrm{Inv}\,\langle X_i | R_i\rangle = \left(X_i \cup X_i^{-1}\right)^+ / \tau_i$ are isomorphic inverse semigroups such that $[c_i\tau_i]_{S_i} \simeq \mathrm{FIS}\left(\{c_i\}\right)$, where $[t]_T$ denotes the inverse subsemigroup of T generated by the element t. Note that the congruences τ_i that appear in the definition of the S_i are *idempotent pure*, and an elegant result of Margolis and Meakin [10] guarantees that the word problem for inverse semigroups presented with idempotent pure congruences is decidable, so we satisfy **(A1)**.

Now we take $U = \mathrm{FIS}(\{c\})$ with the injective maps $\mathrm{w}_i : U \to S_i$ defined on the generator $\{c\}$ as

$$\mathrm{w}_1(c\rho_{\{c\}}) = c_1, \qquad \mathrm{w}_2(c\rho_{\{c\}}) = c_2,$$

and then extended to the entire domain U in the unique way such that the maps $U \ni u \mapsto \mathrm{w}_i(u)\tau_i \in S_i$ are embeddings. It's clear that the injective maps w_i so defined are effectively calculable, so we satisfy **(A2)**.

[5] Remember that N is the number of monochromatic factors of z, i.e. $z = z_{(1)}z_{(2)}...z_{(N)}$.

Given two arbitrary elements $s_1 = z_1\tau_1 \in S_1$ and $s_2 = z_2\tau_2 \in S_2$ (for some $z_i \in \left(X_i \cup X_i^{-1}\right)^+$) it is easy to verify that

$$[z_i\tau_i\!\uparrow_U] = \left\{ \left(c_i^n c_i^{-(n+m)} c_i^m\right)\tau_i \mid z_i\rho_{X_i} \le \left(c_i^n c_i^{-(n+m)} c_i^m\right)\rho_{X_i} \right\},$$

so $[z_i\tau_i\!\uparrow_U]$ is finite and effectively calculable. Moreover it admits a minimum $\mathrm{f}(z_i\tau_i)$ (with respect to the natural partial order $\le$) that is clearly effectively calculable. These remarks are sufficient to prove that we satisfy **(A3)** and **(A4)**.

The last condition **(A5)** follows in analogy with the amalgamated free product of inverse semigroups case, in particular it follows from a results of Cherubini, Meakin and Piochi [4, lemma 3].

It is easy to show that the semigroups S_1 and S_2 have infinite $\mathcal{R}$-classes, for instance we consider $w_0 = a_i a_i^{-1} \in \left(X_i \cup X_i^{-1}\right)^+$ and we define inductively

$$w_{n+1} = a_i w_n b_i b_i^{-1} a_i^{-1}.$$

Now we have a sequence $\{w_n\}_{n\in\mathbb{N}}$ of words in $\left(X_i \cup X_i^{-1}\right)^+$ such that $w_n\tau_i = w_m\tau_i$ for each $n, m \in \mathbb{N}$. Note also that $|\mathrm{V}(\mathrm{MT}_{X_i}(w_n))| > n$ and the Schützenberger automaton $\mathcal{A}(X_i; R_i; w_n\tau_i)$ contains a subtree isomorphic to the Munn tree $\mathrm{MT}_{X_i}(w_m)$ for each $n, m \in \mathbb{N}$, thus the cardinality of $\mathrm{V}(\mathcal{A}(X_i; R_i; w_0\tau_i))$ is not finite.

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References

[1] P. Bennett, *Amalgamated free products of inverse semigroups*, Ph.D. Thesis, Univ. of York, 1994

[2] ———, *Amalgamated free products of inverse semigroups*, Journal of Algebra **198** (1997), 499–534

[3] J.-C. Birget, S. Margolis, J. Meakin, *On the word problem for tensor products and amalgams of monoids*, International J. of Algebra and Computation **9** (1999), 271–294

[4] A. Cherubini, J. Meakin, B. Piochi, *Amalgams of free inverse semigroups*, Semigroup Forum **54** (1997), 199–220

[5] T.E. Hall, *Free products with amalgamation of inverse semigroups*, Journal of Algebra **34** (1975), 375-385

[6] J.M. Howie, **Fundamentals of semigroup theory**, Oxford University Press, 1995

[7] ———, *Amalgamations: a survey*, in Semigroups: algebraic theory and applications to formal languages and codes (ed. C. Bonzini, A. Cherubini and C. Tibiletti) World Scientific, 1994, 125–132

[8] P.R. Jones, S.W. Margolis, J.C. Meakin, J.B. Stephen, *Free products of inverse semigroups II*, Glasgow Math. J. **33** (1991), 373–387

[9] R.C. Lyndon, P.E. Schupp, **Combinatorial Group Theory**, Springer-Verlag, New York, 1977

[10] S.W. Margolis, J.C. Meakin, *Inverse monoids trees and context-free languages*, Trans. Amer. Math. Soc. **335** (1) (1993), 259–276

[11] W.D. Munn, *Free inverse semigroups*, Proc. London Math. Soc. **30** (1974), 385–404

[12] N. Petrich, **Inverse Semigroups**, Wiley, New York, 1984

[13] J.B. Stephen, *Presentation of inverse monoids*, J. Pure Appl. Algebra **63** (1990), 81–112

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