PERIODIC BOUNCE ORBITS OF PRESCRIBED ENERGY

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Abstract. We prove the existence of periodic bounce orbits of prescribed energy on an open bounded domain in $\mathbb{R}^N$. We derive explicit bounds on the period and the number of bounce points.

1. Introduction

Throughout this article we fix an open, bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary and a smooth function $V \in C^\infty(\overline{\Omega})$.[1] We study periodic bounce orbits of the Lagrangian system given by

$$\begin{aligned} L : T\overline{\Omega} = \overline{\Omega} \times \mathbb{R}^N &\to \mathbb{R} \\ (q,v) &\mapsto \tfrac{1}{2}|v|^2 - V(q) \end{aligned} \tag{1.1}$$

that is, continuous and piecewise smooth maps $\gamma : \mathbb{R}/\tau\mathbb{Z} \to \overline{\Omega}$, $\tau > 0$, satisfying the following. There exists a (possibly empty) finite subset $\mathcal{B} \subset [0,\tau]$ such that

(1) γ solves the Euler-Lagrange equation

$$\gamma''(t) + \nabla V(\gamma(t)) = 0 \quad \forall t \notin \mathcal{B} \tag{1.2}$$

(2) for each $t \in \mathcal{B}$ we have $\gamma(t) \in \partial\Omega$, the left resp. right derivatives

$$\gamma'(t^\pm) := \lim_{s \to t^\pm} \gamma'(s) \tag{1.3}$$

exist and γ satisfies the law of reflection

$$\begin{gathered} \big\langle \gamma'(t^+), \nu(\gamma(t)) \big\rangle = -\big\langle \gamma'(t^-), \nu(\gamma(t)) \big\rangle \neq 0\ , \\ \gamma'(t^+) - \big\langle \gamma'(t^+), \nu(\gamma(t)) \big\rangle \cdot \nu(\gamma(t)) = \gamma'(t^-) - \big\langle \gamma'(t^-), \nu(\gamma(t)) \big\rangle \cdot \nu(\gamma(t))\ , \end{gathered} \tag{1.4}$$

where ν is the outer normal to $\partial\Omega$.

Remark 1.1.

- The times $t \in \mathcal{B}$ are called bounce times and $\gamma(t)$ bounce points. In case V is a constant function bounce orbits are billiard trajectories, see [KT91, Tab05] for more details on billiards.
- A periodic bounce orbit with $\mathcal{B} = \emptyset$ is a smooth periodic solution of (1.2).
- For a periodic bounce orbit γ the energy

$$E(\gamma) := \tfrac{1}{2}|\gamma'(t)|^2 + V(\gamma(t)) \tag{1.5}$$

is an integral of motion, namely it is independent of $t \notin \mathcal{B}$.

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[1]We expect that all results remain true if C^∞ is replaced by C^2.

Theorem 1.2. *For all $E > \max_{\overline{\Omega}} V$ there exists a periodic bounce orbit $\gamma : \mathbb{R}/\tau\mathbb{Z} \longrightarrow \overline{\Omega}$ with energy $E(\gamma) = E$, at most $\dim \Omega + 1$ bounce points, and period bounded as follows*

$$\tau \leq C \,\mathrm{diam}(\Omega)\, \frac{\left(E - \min_{\overline{\Omega}} V\right)^{5/2}}{\left(E - \max_{\overline{\Omega}} V\right)^3}\ , \tag{1.6}$$

where C is a constant independent of Ω, V and E (see Propositions 3.2 and 3.7 for an explicit estimate for τ).

Remark 1.3. In dimension two the bound on the number of bounce points in sharp in general. In fact, already for billiard trajectories there are domains $\Omega \subset \mathbb{R}^2$ where every billiard trajectory has at least three bounce points, see for instance [Tab05, Figure 6.6]. It is conceivable that the bound on the number of bounce points is also sharp in higher dimension.

Corollary 1.4. *If in Theorem 1.2 we further require*

$$E(\gamma) > \max_{\overline{\Omega}} V + \tfrac{1}{2}\mathrm{diam}(\overline{\Omega}) \max_{\overline{\Omega}} |\nabla V| \tag{1.7}$$

then the periodic bounce orbit γ has at least one bounce point.

The proofs of Theorem 1.2 and Corollary 1.4 are carried out at the end of Section 3. Inequality (1.6) confirms the physical intuition that there exist orbits whose period decreases as the energy increases. Moreover, asymptotically the minimal period decreases at least as fast as the inverse of the square root of the energy.

Remark 1.5. In their influential work [BG89] Benci-Giannoni prove existence of periodic bounce orbits of prescribed period and with at most $\dim \Omega + 1$ bounce points. This is achieved by studying the classical fixed-time action functional of an approximating smooth Lagrangian system. In this article we replace this by the free-time action functional. Therefore, we detect periodic orbits of prescribed energy rather than period.

A new difficulty in the approximation scheme is to obtain bounds on the periods for approximate solutions independent of the approximation parameter. This is necessary to pass to the limit. To achieve this we employ techniques from symplectic geometry as opposed to the variational techniques used by Benci-Giannoni. This also enables us to give explicit bounds on the period of the periodic bounce orbits in the limit.

We point out that in the case of a constant potential V, say $V \equiv 0$, the result by Benci-Giannoni and the statement of Theorem 1.2 reduce to the mere existence of only one periodic billiard trajectory. In fact, if $V \equiv 0$, given any T-periodic billiard trajectory γ of energy E, the reparametrized curve $\gamma(\frac{\cdot}{\tau})$ is a τT-periodic billiard trajectory of energy $\tau^{-2}E$.

Remark 1.6. Finally, we want to mention two natural generalizations of the set-up considered here. Both seem nontrivial to us, and we will treat them further in future research.

The first generalization is to allow general Riemannian metrics. The approximation scheme can be formulated entirely in Riemannian terms and we are optimistic that is carries over. The same applies to the symplectic topology part. Nevertheless, it is harder to ensure the existence of bouncing points for a sequence of approximating solutions. Indeed, if the Riemannian metric allows a closed geodesic in Ω and the potential V vanishes along such a geodesic then for any energy this closed geodesic (suitably reparametrized) gives a periodic orbit with no bounce points.

Another possible generalization is to add a magnetic field, i.e. "twisting" the symplectic structure on $T^*\overline{\Omega}$ by adding to the canonical symplectic form a closed 2-form σ defined on the base $\overline{\Omega}$. If σ is non-exact then it seems impossible to generalize the methods employed here. First or all, there is no Lagrangian formulation of the problem, in particular, there is no approximation scheme. Second, the approximating energy hypersurfaces in the Hamiltonian formulation may cease to be of contact type, in particular, might be without periodic orbits. Moreover, there is no period-action inequality. These two problems disappear if the magnetic field σ is exact: indeed, in this case, twisting the symplectic form on $T^*\overline{\Omega}$ amounts to adding a primitive of σ to the Hamiltonian while keeping the canonical symplectic structure. Then, there is a Lagrangian formulation and the energy hypersurfaces are of contact type for sufficiently large energy. Nevertheless, the statement of the approximation scheme doesn't readily generalize since near the boundary the magnetic field interacts with the perturbation potential. Also, from a physical point of view one might expect to see "creeping" orbits, that is, orbits which after bouncing are very soon forced back towards the boundary by the magnetic field. Thus, effective bounds on the number of bounce points might be hard to obtain.

Organization of the article. In Section 2 we define the approximation scheme for the free-time action functional and prove that a sequence of approximating solutions converge to periodic bounce orbits of prescribed energy provided their Morse index is uniformly bounded. In Section 3 we study the Hamiltonian formulation of the approximation scheme and prove existence of solutions using techniques from symplectic geometry. Moreover, we derive effective bounds on the period and Morse index. Combining this with the results from Section 2 leads to a proof of Theorem 1.2.

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2. The approximation scheme

Our proof of Theorem 1.2 makes it necessary to modify the beautiful approximation scheme due to Benci-Gianonni [BG89] by replacing the fixed-time action functional by the free-time action functional.

We recall that $\Omega \subset \mathbb{R}^N$ is an open, bounded domain with smooth boundary and $V \in C^\infty(\overline{\Omega})$. We fix $d_0 \in (0, \frac{1}{2})$ sufficiently small, in particular, such that the distance function $\mathrm{dist}_{\partial\Omega}(q) = \min\big\{|q - q'| \;\big|\; q' \in \partial\Omega\big\}$ is smooth at all points $q \in \Omega$ with $\mathrm{dist}_{\partial\Omega}(q) \leq 2d_0$. Let $k : [0,\infty) \to [0, 2d_0]$ be a smooth function such that $0 \leq k' \leq 1$, $k(x) = x$ if $x \leq d_0$ and $k(x) = \mathrm{const}$ if $x \geq 2d_0$. Then, we define a function $h \in C^\infty(\overline{\Omega})$ by

$$h(q) := k(\mathrm{dist}_{\partial\Omega}(q)). \tag{2.1}$$

Notice that h satisfying the following.

- $h(q) = \mathrm{dist}_{\partial\Omega}(q)$ for all $q \in \overline{\Omega}$ with $\mathrm{dist}_{\partial\Omega}(q) \leq d_0$,
- $h(q) > d_0$ if $\mathrm{dist}_{\partial\Omega}(q) > d_0$,
- $0 \leq h \leq 1$ and $h(q) = \mathrm{const}$ if $\mathrm{dist}_{\partial\Omega}(q) \geq 2d_0$,

- $|\nabla h| \leq 1$.

Finally we define a function $U \in C^\infty(\Omega)$ by

$$U(q) := \frac{1}{h^2(q)} \,. \tag{2.2}$$

Thus, U is a positive function that grows like $(\mathrm{dist}_{\partial\Omega})^{-2}$ near $\partial\Omega$ and is constant in the region $\{\mathrm{dist}_{\partial\Omega}(q) \geq 2d_0\}$, see Figure 1.

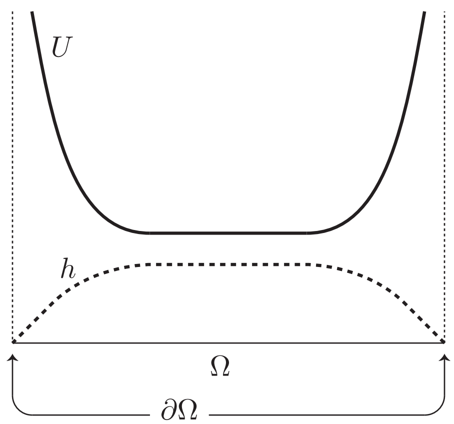

FIGURE 1.

For $\epsilon > 0$, we introduce the modified Lagrangian

$$\begin{aligned} L_\epsilon : T\Omega = \Omega \times \mathbb{R}^N &\to \mathbb{R} \\ (q, v) &\mapsto \tfrac{1}{2}|v|^2 - V(q) - \epsilon U(q) \,. \end{aligned} \tag{2.3}$$

For each energy value $E \in \mathbb{R}$ the free-time action functional $\mathscr{L}_\epsilon^E : H^1(\mathbb{R}/\mathbb{Z}, \overline{\Omega}) \times \mathbb{R}_{>0} \to \mathbb{R}$ is given by

$$\mathscr{L}_\epsilon^E(\Gamma, \tau) := \tau \int_0^1 \Big[L_\epsilon\big(\Gamma(t), \tfrac{1}{\tau}\Gamma'(t)\big) + E \Big] dt \; = \int_0^\tau \Big[L_\epsilon(\gamma, \gamma') + E \Big] dt, \tag{2.4}$$

where $\gamma(t) := \Gamma(\frac{t}{\tau})$. The differential of $\mathscr{L}_\epsilon^E$ is given by

$$\begin{aligned}
d\mathscr{L}_\epsilon^E(\Gamma,\tau)(\Psi,\sigma) &= \tau \int_0^1 \Big[\partial_v L_\epsilon\big(\Gamma, \tfrac{1}{\tau}\Gamma'\big)\tfrac{1}{\tau}\Psi' + \partial_q L_\epsilon\big(\Gamma, \tfrac{1}{\tau}\Gamma'\big)\Psi\Big]dt \\
&\quad + \sigma \int_0^1 \Big[E - \partial_v L_\epsilon\big(\Gamma, \tfrac{1}{\tau}\Gamma'\big)\tfrac{1}{\tau}\Gamma' - L_\epsilon\big(\Gamma, \tfrac{1}{\tau}\Gamma'\big)\Big]dt \\
&= \tau \int_0^1 \Big[\tau^{-2}\langle\Gamma',\Psi'\rangle - \langle\nabla V(\Gamma) + \epsilon\nabla U(\Gamma), \Psi\rangle\Big]dt \\
&\quad + \sigma \int_0^1 \Big[E - (2\tau^2)^{-1}|\Gamma'|^2 + V(\Gamma) + \epsilon U(\Gamma)\Big]dt \\
&= \int_0^\tau \Big[\langle\gamma',\psi'\rangle - \langle\nabla V(\gamma) + \epsilon\nabla U(\gamma), \psi\rangle\Big]dt \\
&\quad + \frac{\sigma}{\tau} \int_0^1 \Big[E - \tfrac{1}{2}|\gamma'|^2 + V(\gamma) + \epsilon U(\gamma)\Big]dt,
\end{aligned} \tag{2.5}$$

where $\psi(t) := \Psi(\frac{t}{\tau})$. Therefore (Γ,τ) is a critical point of $\mathscr{L}_\epsilon^E$ if and only if the corresponding τ-periodic curve γ is a solution of the Euler-Lagrange system

$$\gamma'' + \nabla V(\gamma) + \epsilon\nabla U(\gamma) = 0 \tag{2.6}$$

with energy

$$E_\epsilon(\gamma) := \tfrac{1}{2}|\gamma'(t)|^2 + V(\gamma(t)) + \epsilon U(\gamma(t)) = E\ . \tag{2.7}$$

We prove the analogue of [BG89, Proposition 2.3] for the free-time action functional.

Proposition 2.1. *Let $K > 0$ and $T_2 > T_1 > 0$. For each $\epsilon > 0$, let $(\Gamma_\epsilon, \tau_\epsilon)$ be a critical point of $\mathscr{L}_\epsilon^{E_\epsilon}$ with $T_1 \leq \tau_\epsilon \leq T_2$ and $E_\epsilon \leq K$. Then, up to a subsequence, $(\Gamma_\epsilon,\tau_\epsilon)$ converges to (Γ,τ) in $H^1(S^1,\overline{\Omega}) \times \mathbb{R}_{>0}$ as $\epsilon \to 0$. Moreover, if we define the curve $\gamma(t) := \Gamma(\frac{t}{\tau})$, there exists a finite Borel measure μ on $\mathcal{C} = \{t \in \mathbb{R}/\tau\mathbb{Z} \mid \gamma(t) \in \partial\Omega\}$ such that*

(i) $\displaystyle\int_0^\tau \Big[\langle\gamma',\psi'\rangle - \langle\nabla V(\gamma),\psi\rangle\Big]dt = \int_{\mathcal{C}} \langle\nu(\gamma),\psi\rangle\, d\mu$ *for all $\psi \in H^1(\mathbb{R}/\tau\mathbb{Z};\mathbb{R}^N)$,*

(ii) *γ is a smooth solution of the Euler-Lagrange system of L outside* $\operatorname{supp}(\mu)$, *with energy $E(\gamma) = \lim_{\epsilon\to 0} E_\epsilon$,*

(iii) *γ has left and right derivatives that are left and right continuous on $\mathbb{R}/\tau\mathbb{Z}$ respectively. Moreover, γ satisfies the law of reflection* (1.4) *at each time t which is an isolated point of* $\operatorname{supp}(\mu)$.

In particular, if $\operatorname{supp}(\mu)$ *is a finite set then γ is a periodic bounce orbit of the Lagrangian system given by L and $\mathcal{B} := \operatorname{supp}(\mu)$ is its set of bouncing times.*

Proof. Since the sequences $\{\tau_\epsilon\}$ and $\{E_\epsilon\}$ are bounded, up to a subsequence for $\epsilon \to 0$, we have $\tau_\epsilon \to \tau$ and $E_\epsilon \to E$ with $T_1 \leq \tau \leq T_2$ and $E \leq K$. We show that up to further passing to a subsequence, Γ_ϵ also converges in H^1.

Let $\gamma_\epsilon(t) = \Gamma_\epsilon(\frac{t}{\tau_\epsilon})$ be the periodic orbit corresponding to $(\Gamma_\epsilon,\tau_\epsilon)$. By equations (2.5) and (2.7) we know that the energy $E_\epsilon(\gamma_\epsilon)$ is equal to E_ϵ, and therefore

$$(2\tau_\epsilon)^{-1}|\Gamma_\epsilon'|^2 + V(\Gamma_\epsilon) + \epsilon U(\Gamma_\epsilon) \equiv E_\epsilon\ . \tag{2.8}$$

Moreover, γ_ϵ is a solution of the Euler-Lagrange equation (2.6) associated to L_ϵ, which can be written in terms of $(\Gamma_\epsilon,\tau_\epsilon)$ as

$$\tau_\epsilon^{-2}\Gamma_\epsilon'' + \nabla V(\Gamma_\epsilon) + \epsilon\nabla U(\Gamma_\epsilon) = 0. \tag{2.9}$$

In particular, for each $(\Psi,\sigma)\in H^1(S^1;\mathbb{R}^N)\times\mathbb{R}$ we have $d\mathscr{L}_\epsilon^{E_\epsilon}(\Gamma_\epsilon,\tau_\epsilon)(\Psi,\sigma)=0$ and choosing $\sigma=0$ in equation (2.5) we get

$$\int_0^1 \Big[\tau_\epsilon^{-2}\langle\Gamma_\epsilon',\Psi'\rangle\;-\langle\nabla V(\Gamma_\epsilon),\Psi\rangle\Big]dt=\int_0^1\langle\epsilon\nabla U(\Gamma_\epsilon),\Psi\rangle\,dt,\quad \forall\Psi\in H^1(S^1;\mathbb{R}^N). \tag{2.10}$$

We fix $\Psi=\Psi_\epsilon=-\nabla h(\Gamma_\epsilon)$. By equation (2.8), Γ_ϵ' is uniformly bounded in L^∞, and so is Ψ_ϵ'. Hence, with our choice of Ψ the first two summands on the left hand side of (2.10) are uniformly bounded in ϵ, and thus, so must be the third summand, i.e.

$$\int_0^1\langle\epsilon\nabla U(\Gamma_\epsilon),\Psi_\epsilon\rangle\,dt=\int_0^1\frac{2\epsilon}{h^3(\Gamma_\epsilon)}|\nabla h(\Gamma_\epsilon)|^2\,dt\le C. \tag{2.11}$$

Let $\Omega'\subset\Omega$ be the compact neighborhood of $\partial\Omega$ given by

$$\Omega'=\{q\in\Omega\,|\,h(q)\le d_0\}, \tag{2.12}$$

where d_0 is the positive constant that enters the definition of the function h. Notice that on Ω' we have $h=\mathrm{dist}_{\partial\Omega}$ and in particular $|\nabla h|=1$. Moreover, on $\Omega\setminus\Omega'$ we have $h>d_0$ and $|\nabla h|\le 1$. These properties, together with the estimate (2.11), give the uniform bound

$$\int_0^1\frac{2\epsilon}{h^3(\Gamma_\epsilon)}\,dt\le\int_0^1\frac{2\epsilon}{h^3(\Gamma_\epsilon)}|\nabla h(\Gamma_\epsilon)|^2\,dt+\frac{2\epsilon}{d_0^3}\le C+\frac{2\epsilon}{d_0^3}\,. \tag{2.13}$$

This proves that $\epsilon\nabla U(\Gamma_\epsilon)$ is uniformly bounded in L^1 because

$$\epsilon\nabla U(\Gamma_\epsilon)=-\frac{2\epsilon}{h^3(\Gamma_\epsilon)}\nabla h(\Gamma_\epsilon) \tag{2.14}$$

and $|\nabla h|\le 1$. Since $\nabla V(\Gamma_\epsilon)$ is also uniformly bounded in L^1 (actually in L^∞), the Euler-Lagrange equation (2.9) together with $T_1\le\tau_\epsilon\le T_2$ forces Γ_ϵ'' to be uniformly bounded in L^1 as well. Thus, Γ_ϵ is uniformly bounded in $W^{2,1}$. By the compactness of the embedding $W^{2,1}(S^1;\mathbb{R}^N)\hookrightarrow H^1(S^1;\mathbb{R}^N)$, up to passing to a subsequence for $\epsilon\to0$, we have that Γ_ϵ converges to some $\Gamma:S^1\to\overline{\Omega}$ in H^1.

Now, since the functions $\tilde\mu_\epsilon:=2\epsilon h^{-3}(\Gamma_\epsilon)$ are uniformly bounded in L^1, up to passing to a subsequence for $\epsilon\to0$, $\tilde\mu_\epsilon$ converges to some $\tilde\mu$ in L^1 weak-$*$. By the Riesz representation Theorem, $\tilde\mu$ is a positive, finite Borel measure. We set

$$\mathcal{C}':=\{t\in S^1\,|\,\Gamma(t)\in\partial\Omega\}\,. \tag{2.15}$$

Since, for each $t\notin\mathcal{C}'$, the function $\tilde\mu_\epsilon$ converges uniformly to 0 in a neighborhood of t the support of $\tilde\mu$ is contained in $\mathcal{C}'$. Moreover, if $t\in\mathcal{C}'$, for $\epsilon\to0$ the sequence $\nabla h(\Gamma_\epsilon(t))$ converges to $-\nu(\Gamma(t))$. Thus, taking the limit $\epsilon\to0$ in (2.10) we obtain

$$\tau^{-2}\int_0^1\langle\Gamma',\Psi'\rangle\,dt-\int_0^1\langle\nabla V(\Gamma),\Psi\rangle\,dt=\int_{\mathcal{C}'}\langle\nu(\Gamma),\Psi\rangle\,d\tilde\mu,\qquad\forall\Psi\in H^1(S^1;\mathbb{R}^N). \tag{2.16}$$

By the reparametrization $\mathbb{R}/\tau\mathbb{Z}\to S^1$ given by $t\mapsto\frac{t}{\tau}$ the measure $\tilde\mu$ is pulled-back to a measure μ on $\mathcal{C}:=\{t\in\mathbb{R}/\tau\mathbb{Z}\,|\,\gamma(t)\in\partial\Omega\}$ and the above equation can be rewritten as in point (i) of the statement.

Now, if $t\notin\mathrm{supp}(\mu)$, we can take $\epsilon>0$ sufficiently small such that $[t-\epsilon,t+\epsilon]\cap\mathrm{supp}(\mu)=\emptyset$. For each $\psi\in H^1(\mathbb{R}/\tau\mathbb{Z};\mathbb{R}^N)$ supported in $[t-\epsilon,t+\epsilon]$, point (i) reduces to

$$\int_{t-\epsilon}^{t+\epsilon}\Big[\langle\gamma',\psi'\rangle-\langle\nabla V(\gamma),\psi\rangle\Big]dt=0, \tag{2.17}$$

and a usual bootstrap argument readily implies that γ is a smooth solution of the Euler-Lagrange equation of L on $[t-\epsilon, t+\epsilon]$. This establishes point (ii).

Now, point (i) also implies that γ' is a curve of bounded variation. Therefore γ has left and right derivatives at each point and they are left and right continuous respectively. In order to conclude the proof, we only need to establish that the reflection rule is satisfied at each time $t \in \operatorname{supp}(\mu)$.

Up to passing to a subsequence for $\epsilon \to 0$, the sequence $\epsilon U(\gamma_\epsilon)$ converges to 0 almost everywhere. Indeed, assume that $\epsilon U(\gamma_\epsilon)$ does not converge to zero on a set $I \subset \mathbb{R}/\tau\mathbb{Z}$. Then, $h(\gamma_\epsilon) \to 0$ and $|\nabla h(\gamma_\epsilon)| \to 1$ pointwise on I. Since

$$\epsilon \nabla U(\gamma_\epsilon) = \epsilon U(\gamma_\epsilon) \frac{-2\nabla h(\gamma_\epsilon(t))}{h(\gamma_\epsilon(t))}\ , \tag{2.18}$$

then $|\epsilon \nabla U(\gamma_\epsilon)| \to +\infty$ pointwise on I. Now, assume that I has positive Lebesgue measure. By Fatou's Lemma we get

$$\liminf_{\epsilon\to 0} \int_I |\epsilon \nabla U(\gamma_\epsilon)|\, dt \geq \int_I \liminf_{\epsilon \to 0} |\epsilon \nabla U(\gamma_\epsilon)|\, dt = +\infty, \tag{2.19}$$

which contradicts the fact that $\epsilon \nabla U(\gamma_\epsilon)$ is uniformly bounded in L^1.

Since $\epsilon U(\gamma_\epsilon)$ converges to 0 almost everywhere and $E_\epsilon \to E$, we have that $\frac{1}{2}|\gamma'| + V(\gamma) = E$ almost everywhere, and since γ' has bounded variation we actually obtain

$$\tfrac{1}{2}|\gamma'(t^\pm)| + V(\gamma(t)) = E \qquad \forall t \in \mathbb{R}/\tau\mathbb{Z}. \tag{2.20}$$

Now, let us consider a time t which is an isolated point in $\operatorname{supp}(\mu)$. In point (i) of the statement, let us choose ψ to be supported in the interval $[t-\epsilon, t+\epsilon]$, where $\epsilon > 0$ is sufficiently small so that $[t-\epsilon, t+\epsilon] \cap \operatorname{supp}(\mu) = \{t\}$. After an integration by parts we obtain

$$\langle \gamma'(t^-) - \gamma'(t^+), \psi(t)\rangle - \int_{[t-\epsilon,t+\epsilon]\setminus\{t\}} \langle \gamma'' + \nabla V(\gamma), \psi\rangle\, dt = \langle \nu(\gamma(t)), \psi(t)\rangle\, \mu(\{t\})\,. \tag{2.21}$$

Since γ is a solution of the Euler-Lagrange equation of L on $[t-\epsilon, t+\epsilon]\setminus\{t\}$, the integral on the left-hand side is zero and we actually have

$$\langle \gamma'(t^-) - \gamma'(t^+), v\rangle = \langle \nu(\gamma(t)), v\rangle\, \mu(\{t\}), \qquad \forall v \in \mathbb{R}^N\,. \tag{2.22}$$

Choosing v to be an arbitrary vector tangent to $\partial\Omega$ at $\gamma(t)$, namely $\langle \nu(\gamma(t)), v\rangle = 0$, we obtain that the components of $\gamma'(t^-)$ and $\gamma'(t^+)$ tangent to $\partial\Omega$ are the same, i.e.

$$\gamma'(t^+) - \langle \nu(\gamma(t)), \gamma'(t^+)\rangle \cdot \nu(\gamma(t)) = \gamma'(t^-) - \langle \nu(\gamma(t)), \gamma'(t^-)\rangle \cdot \nu(\gamma(t))\,. \tag{2.23}$$

This, together with conservation of energy (2.20), implies that

$$|\langle \nu(\gamma(t)), \gamma'(t^+)\rangle| = |\langle \nu(\gamma(t)), \gamma'(t^-)\rangle|, \tag{2.24}$$

and if this latter quantity is nonzero then we must have

$$\langle \nu(\gamma(t)), \gamma'(t^+)\rangle = -\langle \nu(\gamma(t)), \gamma'(t^-)\rangle. \tag{2.25}$$

Finally, by choosing $v = \nu(\gamma(t))$ in equation (2.22) we obtain

$$\langle \nu(\gamma(t)), \gamma'(t^+)\rangle = \tfrac{1}{2}\langle \gamma'(t^-) - \gamma'(t^+), \nu(\gamma(t))\rangle = \tfrac{1}{2}\mu(\{t\}) \neq 0. \tag{2.26}$$

This concludes the proof of point (iii). □

Proposition 2.2. *We consider the situation of Proposition 2.1. Then, up to taking a subsequence of* $\{(\Gamma_\epsilon, \tau_\epsilon)\}$, *the cardinality* $|\mathrm{supp}(\mu)|$ *of the support of the measure* μ *is bounded from above by the Morse index of the restricted functional* $\mathscr{L}_\epsilon^{E_\epsilon}|_{H^1\times\{\tau_\epsilon\}}$ *at* Γ_ϵ *for all* ϵ *sufficiently small, i.e.*

$$|\mathrm{supp}(\mu)| \leq \liminf_{\epsilon\to 0} \mu_{\mathrm{Morse}}\big(\Gamma_\epsilon; \mathscr{L}_\epsilon^{E_\epsilon}|_{H^1\times\{\tau_\epsilon\}}\big) . \tag{2.27}$$

Proof. With the notation adopted in the proof of Proposition 2.1 (see in particular the paragraph of equation (2.15)), the measure μ is the pullback of a measure $\tilde\mu$ on $S^1 = \mathbb{R}/\mathbb{Z}$ via the reparametrization $\iota : \mathbb{R}/\tau\mathbb{Z} \to S^1$ given by $\iota(t) = \frac{t}{\tau}$. In particular $\iota(\mathrm{supp}(\mu)) = \mathrm{supp}(\tilde\mu)$ and

$$|\mathrm{supp}(\mu)| = |\mathrm{supp}(\tilde\mu)|. \tag{2.28}$$

Hence, in order to prove the proposition it is enough to establish the following: for each point $t \in \mathrm{supp}(\tilde\mu)$ and for each $\epsilon > 0$ sufficiently small, there exists a vector field $\Psi_\epsilon \in H^1(S^1;\mathbb{R}^N)$ supported on an sufficiently small neighborhood of t and such that

$$\mathrm{Hess}\mathscr{L}_\epsilon^{E_\epsilon}(\Gamma_\epsilon, \tau_\epsilon)[(\Psi_\epsilon, 0), (\Psi_\epsilon, 0)] < 0. \tag{2.29}$$

In fact, assume that this is verified. Then, for k distinct points $t_1, ..., t_k \in \mathrm{supp}(\tilde\mu)$ and sufficiently small $\epsilon > 0$ we can find k vector fields $\Psi_{\epsilon,1}, ..., \Psi_{\epsilon,k}$ such that each $\Psi_{\epsilon,j}$ is supported in a sufficiently small neighborhood of t_j and verifies (2.29). In particular, we may assume that the supports of the $\Psi_{\epsilon,j}$'s are pairwise disjoint. Therefore, these vector fields span a k-dimensional vector subspace of $H^1(S^1;\mathbb{R}^N)$ over which the Hessian of the restricted action functional[2] $\mathscr{L}_\epsilon^{E_\epsilon}|_{H^1\times\{\tau_\epsilon\}}$ at Γ_ϵ is negative definite, which implies

$$\mu_{\mathrm{Morse}}\big(\Gamma_\epsilon; \mathscr{L}_\epsilon^{E_\epsilon}|_{H^1\times\{\tau_\epsilon\}}\big) \geq k. \tag{2.30}$$

Let us now establish the assertion made at the beginning of the proof. From now on we fix $t \in \mathrm{supp}(\tilde\mu)$ and $\epsilon > 0$ sufficiently small. For $\delta > \delta' > 0$ we choose a smooth function $\phi_\epsilon : \mathbb{R}/\mathbb{Z} \to [0, 1]$ such that $\mathrm{supp}(\phi_\epsilon) \subseteq [t - \delta, t + \delta]$ and $\phi_\epsilon \equiv 1$ on $[t - \delta', t + \delta']$. We define the vector field $\Psi_\epsilon \in H^1(S^1;\mathbb{R}^N)$ by

$$\Psi_\epsilon(s) := -\phi_\epsilon(s)\nabla h(\Gamma_\epsilon(s)). \tag{2.31}$$

We will show that Ψ_ϵ satisfies (2.29). The left-hand side of (2.29) computes to

$$\begin{aligned}
&\mathrm{Hess}\mathscr{L}_\epsilon^{E_\epsilon}(\Gamma_\epsilon, \tau_\epsilon)[(\Psi_\epsilon, 0), (\Psi_\epsilon, 0)] \\
&\quad= \tau_\epsilon \int_0^1 \Big[\tau_\epsilon^{-2}\langle \partial_{vv} L_\epsilon(\Gamma_\epsilon, \tfrac{1}{\tau_\epsilon}\Gamma_\epsilon')\Psi_\epsilon', \Psi_\epsilon'\rangle + \langle \partial_{qq} L_\epsilon(\Gamma_\epsilon, \tfrac{1}{\tau_\epsilon}\Gamma_\epsilon')\Psi_\epsilon, \Psi_\epsilon\rangle\Big] ds \\
&\quad= A_\epsilon - B_\epsilon,
\end{aligned} \tag{2.32}$$

where

$$\begin{aligned}
A_\epsilon &= \tau_\epsilon \int_0^1 \left[\tau_\epsilon^{-2}|\Psi_\epsilon'|^2 - \langle \nabla^2 V(\Gamma_\epsilon)\Psi_\epsilon, \Psi_\epsilon\rangle + 2\epsilon \frac{\langle \nabla^2 h(\Gamma_\epsilon)\Psi_\epsilon, \Psi_\epsilon\rangle}{h^3(\Gamma_\epsilon)}\right] dt, \\
B_\epsilon &= 6\tau_\epsilon\epsilon \int_0^1 \frac{\langle \nabla h(\Gamma_\epsilon), \Psi_\epsilon\rangle^2}{h^4(\Gamma_\epsilon)}\, ds.
\end{aligned} \tag{2.33}$$

Now, the term $|A_\epsilon|$ is uniformly bounded in ϵ. Indeed, since Γ_ϵ converges in H^1, the vector field Ψ_ϵ is uniformly bounded in H^1, which implies that Ψ_ϵ' is uniformly bounded in L^2. Moreover, in the proof of Proposition 2.1 (see equation (2.13)) we showed that $2\epsilon h^{-3}(\Gamma_\epsilon)$ is

[2]Notice that $\mathrm{Hess}\mathscr{L}_\epsilon^{E_\epsilon}|_{H^1\times\{\tau_\epsilon\}}(\Gamma_\epsilon)[\Psi, \Xi] = \mathrm{Hess}\mathscr{L}_\epsilon^{E_\epsilon}(\Gamma_\epsilon, \tau_\epsilon)[(\Psi, 0), (\Xi, 0)]$.

uniformly bounded in L^1, and therefore the last summand under the integral in A_ϵ is also uniformly bounded in L^1.

As for B_ϵ, we want to show that it goes to $+\infty$ as $\epsilon \to 0$. Since $\Gamma_\epsilon \to \Gamma$ in H^1 (in particular in C^0) as $\epsilon \to 0$ and $|\nabla h| = 1$ on $\partial\Omega$, we can find $\delta'' \in (0, \delta']$ such that $|\nabla h(\Gamma_\epsilon(s))|^4 \geq \frac{1}{2}$ for each $s \in [t - \delta'', t + \delta'']$ and $\epsilon > 0$ sufficiently small. Therefore we can estimate

$$
\begin{aligned}
B_\epsilon &\geq 6T_1\epsilon \int_{t-\delta''}^{t+\delta''} \frac{\langle \nabla h(\Gamma_\epsilon), \Psi_\epsilon\rangle^2}{h^4(\Gamma_\epsilon)}\, ds \\
&= 6T_1\epsilon \int_{t-\delta''}^{t+\delta''} \frac{|\nabla h(\Gamma_\epsilon)|^4}{h^4(\Gamma_\epsilon)}\, ds \\
&\geq \frac{6T_1\epsilon}{2} \int_{t-\delta''}^{t+\delta''} \frac{1}{h^4(\Gamma_\epsilon)}\, ds \\
\text{(by Hölder ineq.)} &\geq \frac{6T_1\epsilon}{2(2\delta'')^{1/3}} \left(\int_{t-\delta''}^{t+\delta''} \frac{1}{h^3(\Gamma_\epsilon)}\, ds \right)^{4/3} \\
&= \frac{6T_1}{2(2\delta'')^{1/3}} \underbrace{\left(\int_{t-\delta''}^{t+\delta''} \frac{\epsilon}{h^3(\Gamma_\epsilon)}\, ds \right)}_{=:\, B'_\epsilon} \underbrace{\left(\int_{t-\delta''}^{t+\delta''} \frac{1}{h^3(\Gamma_\epsilon)}\, ds \right)^{1/3}}_{=:\, B''_\epsilon}.
\end{aligned}
\tag{2.34}
$$

As we showed in the proof of Proposition 2.1 (see the paragraph of equation (2.15)), up to a subsequence the function $2\epsilon h^{-3}(\Gamma_\epsilon)$ converges to the measure $\tilde{\mu}$ in L^1 weak-$*$, which implies that B'_ϵ converges to a constant $B' \geq \frac{1}{2}\tilde{\mu}(\{t\}) > 0$. Hence, it remains to be shown that $B''_\epsilon \to +\infty$ as $\epsilon \to 0$. By the Hölder inequality we get

$$
B''_\epsilon = \int_{t-\delta''}^{t+\delta''} \frac{1}{h^3(\Gamma_\epsilon)}\, ds \geq (2\delta'')^{-1/2} \underbrace{\left(\int_{t-\delta''}^{t+\delta''} \frac{1}{h^2(\Gamma_\epsilon)}\, ds \right)^{3/2}}_{=:\, B'''_\epsilon}. \tag{2.35}
$$

We recall that up to a subsequence $\Gamma_\epsilon \to \Gamma$ as $\epsilon \to 0$ in H^1, and that $\Gamma(t) \in \partial\Omega$. By the definition of h, if we choose δ'' small enough we have that

$$
h(\Gamma_\epsilon(s)) = \mathrm{dist}_{\partial\Omega}(\Gamma_\epsilon(s)), \qquad \forall s \in [t - \delta'', t + \delta'']. \tag{2.36}
$$

for all $\epsilon > 0$ sufficiently small. Then, let $D > 0$ be a uniform upper bound for the L^2 norm of the vector fields Γ'_ϵ. For each $s \in [t - \delta'', t + \delta'']$ we can estimate using $|\nabla h| \leq 1$

$$
|h(\Gamma_\epsilon(s)) - h(\Gamma_\epsilon(t))| \leq |\Gamma_\epsilon(s) - \Gamma_\epsilon(t)| \leq |s - t|^{1/2} \|\Gamma'_\epsilon\|_{L^2} \leq |s - t|^{1/2} D. \tag{2.37}
$$

This implies

$$
\begin{aligned}
B'''_\epsilon &= \int_{t-\delta''}^{t+\delta''} \frac{1}{h^2(\Gamma_\epsilon(s))}\, ds \geq \int_{t-\delta''}^{t+\delta''} \frac{1}{(h(\Gamma_\epsilon(0)) + |s - t|^{1/2} D)^2}\, ds \\
&= \int_{-\delta''}^{\delta''} \frac{1}{(h(\Gamma_\epsilon(0)) + |s|^{1/2} D)^2}\, ds = 2\int_0^{\delta''} \frac{1}{(h(\Gamma_\epsilon(0)) + s^{1/2} D)^2}\, ds \\
&\geq \int_0^{\delta''} \frac{1}{h^2(\Gamma_\epsilon(0)) + sD^2}\, ds = \frac{1}{D^2} \ln\left(1 + \frac{D^2\delta''}{h(\Gamma_\epsilon(0))} \right).
\end{aligned}
\tag{2.38}
$$

Since up to a subsequence for $\epsilon \to 0$ we have $h(\Gamma_\epsilon(0)) \to 0$, from the above estimate we infer that $B'''_\epsilon \to +\infty$. Thus, this shows that $B_\epsilon \to +\infty$ and therefore the proposition follows. □

Next, we examine the case in Proposition 2.1 where the periods go to zero.

Proposition 2.3. *Let $K>0$ and $(\Gamma_\epsilon,\tau_\epsilon)$ be a critical point of $\mathscr{L}_\epsilon^{E_\epsilon}$ with $E_\epsilon \le K$ and $\tau_\epsilon \to 0$ as $\epsilon \to 0$. Then, up to a subsequence for $\epsilon \to 0$, Γ_ϵ converges in C^0 to a constant curve $\gamma \equiv q \in \overline{\Omega}$. Moreover, one of the following holds.*

- (i) *q is a critical point of the potential V.*
- (ii) *q lies in $\partial\Omega$ and there exists $a>0$ such that $\nabla V(q) = -a\nu(q)$, where ν is the outer normal to $\partial\Omega$.*

Remark 2.4. In case (ii) of Proposition 2.3 the stationary curve $\gamma(t)\equiv q$ describes a particle confined by the potential, see Figure 2.

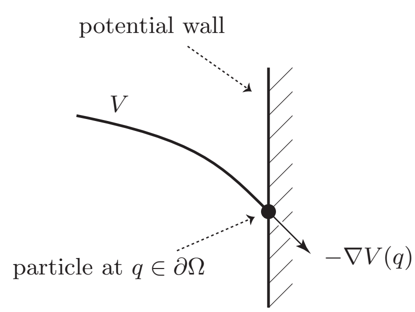

Figure 2.

Proof. We choose a sequence of positive integers $\{\kappa_\epsilon\}$ such that $T_1 < \kappa_\epsilon\tau_\epsilon < T_2$ for suitable $T_2 > T_1 > 0$ and we define $(\Theta_\epsilon,\sigma_\epsilon) \in H^1(S^1;\overline{\Omega}) \times \mathbb{R}_{>0}$ by $\Theta_\epsilon(t) := \Gamma_\epsilon(\kappa_\epsilon t)$ and $\sigma_\epsilon := \kappa_\epsilon\tau_\epsilon$. We point out that $(\Theta_\epsilon,\sigma_\epsilon)$ is a critical point of the action functional $\mathscr{L}_\epsilon^{E_\epsilon}$. By Proposition 2.1 we conclude that, up to a subsequence, $(\Theta_\epsilon,\sigma_\epsilon) \to (\Theta,\sigma)$ in $H^1(S^1,\overline{\Omega}) \times \mathbb{R}_{>0}$ as $\epsilon \to 0$. In particular $\Theta_\epsilon \to \Theta$ in C^0.

We claim that Θ is a constant curve. Indeed, let us assume by contradiction that there exist $t_1 < t_2$ such that

$$|\Theta(t_1) - \Theta(t_2)| > 0 \tag{2.39}$$

Notice that each Θ_ϵ is ${\kappa_\epsilon}^{-1}$ periodic, and in particular

$$\Theta_\epsilon(t_2) = \Theta_\epsilon(t_2 - j\kappa_\epsilon^{-1}), \qquad \forall j \in \mathbb{N}. \tag{2.40}$$

Since $\kappa_\epsilon \to \infty$, we can find a sequence of positive integers $\{j_\epsilon\}$ such that $j_\epsilon\kappa_\epsilon^{-1} \to t_2 - t_1$. This, together with the C^0 convergence $\Theta_\epsilon \to \Theta$, implies

$$\Theta(t_1) = \lim_{\epsilon\to 0}\Theta_\epsilon(t_1) = \lim_{\epsilon\to 0}\Theta_\epsilon(t_2 - j_\epsilon\kappa_\epsilon^{-1}) = \lim_{\epsilon\to 0}\Theta_\epsilon(t_2) = \Theta(t_2), \tag{2.41}$$

which contradicts (2.39).

Since each curve Θ_ϵ is an iteration of Γ_ϵ, the fact that Θ_ϵ converges in C^0 to a constant curve forces Γ_ϵ to converge in C^0 to the same constant curve $\Gamma = \Theta \equiv q \in \overline{\Omega}$. Then, the integral equation in point (i) of Proposition 2.1 reduces to

$$-\int_0^\sigma \langle \nabla V(q), \psi\rangle\, dt = \int_{\mathcal{C}} \langle \nu(q), \psi\rangle\, d\mu \qquad \forall \psi \in C^\infty(\mathbb{R}/\sigma\mathbb{Z};\mathbb{R}^N). \tag{2.42}$$

Here, $\mathcal{C} = \emptyset$ if $q \in \Omega$ and $\mathcal{C} = \mathbb{R}/\sigma\mathbb{Z}$ if $q \in \partial\Omega$. This immediately implies the proposition. □

3. Proof of Theorem 1.2

In order to prove Theorem 1.2, it only remains to build the sequence $\{(\Gamma_\epsilon, \tau_\epsilon)\}$ of critical points of the free-time action functionals $\{\mathscr{L}_\epsilon^{E_\epsilon}\}$, as needed in Proposition 2.1. This will be carried out in the Hamiltonian formulation, by considering the Hamiltonian function

$$\begin{aligned} H_\epsilon : T^*\Omega = \Omega \times \mathbb{R}^N &\to \mathbb{R} \\ (q,p) &\mapsto \tfrac{1}{2}|p|^2 + V(q) + \epsilon U(q)\ . \end{aligned} \tag{3.1}$$

This Hamiltonian is the Legendre-dual to the Lagrangian L_ϵ defined in (2.3). By Legendre duality, τ-periodic Hamiltonian orbits $v : \mathbb{R}/\tau\mathbb{Z} \to \Omega \times \mathbb{R}^N$ of H_ϵ with energy $H_\epsilon(v) = E$ are in one-to-one correspondence to τ-periodic solutions $\gamma = \pi(v)$ of the Euler-Lagrange system of L_ϵ with energy $E_\epsilon(\gamma) = E$ via the projection $\pi : T^*\Omega \to \Omega$.

We begin with the following

Lemma 3.1. *Any energy value $E > \max_{\overline{\Omega}} V$ is a regular value of the Hamiltonian function H_ϵ provided $\epsilon > 0$ is sufficiently small.*

Proof. Since H_ϵ is a classical Hamiltonian (i.e. of the form kinetic energy plus potential), the energy hypersurface Σ_ϵ is regular provided the boundary of its projection into the base, i.e. the set

$$\Upsilon_\epsilon = \partial\,\pi(\Sigma_\epsilon) = \{V + \epsilon U = E\} \subset \mathbb{R}^N\,, \tag{3.2}$$

does not contain any critical point of the potential $V + \epsilon U$. This is always verified if ϵ is sufficiently small. Indeed, for $q \in \Upsilon_\epsilon$ we have by (2.2)

$$h^2(q) = \frac{\epsilon}{E - V(q)}\,, \tag{3.3}$$

and therefore

$$\begin{aligned} |\nabla V(q) + \epsilon\nabla U(q)| &\geq |\epsilon\nabla U(q)| - |\nabla V(q)| \\ &= \frac{2\epsilon}{h^3(q)}\,|\nabla h(q)| - |\nabla V(q)| \\ &= 2\epsilon^{-1/2}(E - V(q))^{3/2}|\nabla h(q)| - |\nabla V(q)| \\ &\geq 2\epsilon^{-1/2}(E - \max_{\overline{\Omega}} V)^{3/2}|\nabla h(q)| - |\nabla V(q)|\,. \end{aligned} \tag{3.4}$$

Equation (3.3) implies that $h|_{\Upsilon_\epsilon} \to 0$ uniformly as $\epsilon \to 0$. Hence, for sufficiently small ϵ we have $h|_{\Upsilon_\epsilon} = \mathrm{dist}_{\partial\Omega}|_{\Upsilon_\epsilon}$ and $|\nabla h| \geq \frac{1}{2}$ on Υ_ϵ. Combining this with (3.4) we obtain

$$|\nabla V(q) + \epsilon\nabla U(q)| \geq \epsilon^{-1/2}(E - \max_{\overline{\Omega}} V)^{3/2} - |\nabla V(q)|, \qquad \forall q \in \Upsilon_\epsilon, \tag{3.5}$$

from which we conclude that $\nabla V + \epsilon\nabla U$ does not vanish on Υ_ϵ for ϵ sufficiently small. □

From now on we fix an energy value $E > \max_{\overline{\Omega}} V$ and we consider $\epsilon > 0$ small enough so that Lemma 3.1 holds. In particular, the energy hypersurface

$$\Sigma_\epsilon := \{H_\epsilon = E\} \tag{3.6}$$

is a smooth and non-empty closed manifold. Notice that $\pi : T^*\Omega \to \Omega$ projects Σ_ϵ into the compact set $\Omega_\epsilon := \{\epsilon U \leq E\}$. Thus, we can modify the potential ϵU outside Ω_ϵ and extend it to a global potential $U_\epsilon \in C^\infty(\mathbb{R}^N)$ such that $U_\epsilon = \epsilon U$ on $\Omega_{\epsilon/2}$, $U_\epsilon > E$ outside $\Omega_{\epsilon/2}$ and $U \equiv E' > E$ outside Ω, see Figure 3. Analogously, we extend V to a compactly supported

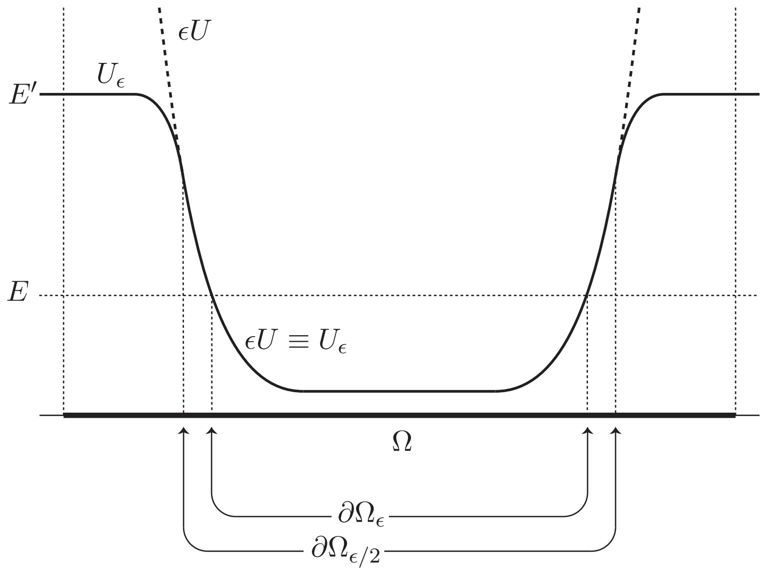

FIGURE 3. The potential ϵU and its modification U_ϵ.

function $V \in C^\infty(\mathbb{R}^N)$ such that $V > -(E' - E)$. In particular $V + U_\epsilon > E$ outside Ω_ϵ, and $V + U_\epsilon \equiv E'$ outside a compact neighborhood of $\overline{\Omega}$.

For technical reasons we compactify $\mathbb{R}^N$ to S^N in such a way that

$$\mathrm{vol}(S^N) \gg \mathrm{vol}(\Omega), \tag{3.7}$$

and we further extend U_ϵ and V to smooth functions on S^N that we still denote by U_ϵ and V. Finally, we introduce the modified Hamiltonian

$$\begin{aligned} K_\epsilon : T^*S^N &\to \mathbb{R} \\ (q,p) &\mapsto \tfrac{1}{2}|p|^2 + V(q) + U_\epsilon(q)\ . \end{aligned} \tag{3.8}$$

Notice that $\Sigma_\epsilon = \{K_\epsilon = E\}$ and the Hamiltonian flows of H_ϵ and K_ϵ agree on Σ_ϵ.

Since K_ϵ is a classical Hamiltonian a well-known result in Hamiltonian dynamics asserts that the energy hypersurface Σ_ϵ is of restricted contact type, i.e. there exists a primitive λ_ϵ of the canonical symplectic form ω of T^*S^N such that $\lambda_\epsilon|_{\Sigma_\epsilon}$ is a contact form. We recall that a primitive λ_ϵ of ω restricts to a contact form on Σ_ϵ if and only if the associated Liouville vector field P_ϵ, defined by $\omega(P_\epsilon, \cdot) = \lambda_\epsilon$, is transverse to Σ_ϵ. This is equivalent to asking that $\lambda_\epsilon(X_\epsilon) \neq 0$, since

$$\lambda_\epsilon(X_\epsilon) = \omega(P_\epsilon, X_\epsilon) = dK_\epsilon(P_\epsilon). \tag{3.9}$$

For later purposes, we need to show that we can choose λ_ϵ such that $\lambda_\epsilon(X_\epsilon)$ is bounded away from zero uniformly in ϵ.

Proposition 3.2. *We fix $E > \max_{\overline{\Omega}} V$. For $\epsilon > 0$ small enough there exists a 1-form λ_ϵ on T^*S^N with $d\lambda_\epsilon = \omega$ which restricts to a contact form on $\Sigma_\epsilon = \{K_\epsilon = E\}$. Moreover, on Σ_ϵ we have the estimate*

$$\lambda_\epsilon(X_\epsilon) \geq \frac{(E - \max_{\overline{\Omega}} V)^3}{2\big[(E - \max_{\overline{\Omega}} V)^2 + 48(E - \min_{\overline{\Omega}} V)^2\big]} =: \Lambda(E) > 0\ . \tag{3.10}$$

PROOF. We denote by $\lambda = \sum_i p_i dq_i$ the Liouville 1-form on T^*S^N. The Hamiltonian vector field X_ϵ of K_ϵ is given in local coordinates by

$$X_\epsilon = \sum_i \left[p_i \frac{\partial}{\partial q_i} - \left(\frac{\partial U_\epsilon}{\partial q_i} + \frac{\partial V}{\partial q_i} \right) \frac{\partial}{\partial p_i} \right]. \tag{3.11}$$

Thus, we have $\lambda(X_\epsilon) = |p|^2 \geq 0$. Now we consider $u : T^*\Omega \to \mathbb{R}$ given by

$$u(q,p) = dU(q)[p] = \sum_i \frac{\partial U}{\partial q_i}(q) p_i \tag{3.12}$$

and define the 1-form

$$\lambda_\epsilon := \lambda - C\epsilon\, du = \lambda - C\epsilon \sum_{i,j} \frac{\partial^2 U_\epsilon}{\partial q_i\, \partial q_j}\, p_i\, dq_j - C\epsilon \sum_i \frac{\partial U_\epsilon}{\partial q_i}\, dp_i, \tag{3.13}$$

where $C > 0$ is a constant independent of ϵ that we will fix later. Since $U_\epsilon = \epsilon U$ on Σ_ϵ and using the definition of U (see (2.2)), the function $\lambda_\epsilon(\Sigma_\epsilon)$ on Σ_ϵ is given by

$$\begin{aligned} \lambda_\epsilon(X_\epsilon)\big|_{\Sigma_\epsilon} &= |p|^2 - C\epsilon^2\, \mathrm{Hess}U(q)[p,p] + C\epsilon^3\, |\nabla U|^2 + C\epsilon^2\, \langle \nabla U, \nabla V \rangle \\ &= |p|^2 + 2C\epsilon^2 h^{-3}\, \mathrm{Hess}h(q)[p,p] + 4C\epsilon^3 h^{-6}\, |\nabla h|^2 \\ &\quad - 6C\epsilon^2 h^{-4}\, |dh(q)p|^2 - 2C\epsilon^2 h^{-3} \langle \nabla h, \nabla V \rangle. \end{aligned} \tag{3.14}$$

Now we notice that for $(q,p) \in \Sigma_\epsilon$ we have

$$h^2(q) = \frac{\epsilon}{E - V(q) - \frac{1}{2}|p|^2}\,. \tag{3.15}$$

We choose $\kappa \geq 0$ such that

$$\mathrm{Hess}h(q)[p,p] \geq -\kappa |p|^2 \quad \forall q \in \Omega. \tag{3.16}$$

Then using $|\nabla h| \leq 1$ we have the estimate

$$\begin{aligned} \lambda_\epsilon(X_\epsilon)\big|_{\Sigma_\epsilon} &\geq |p|^2 \big(1 - 6C\epsilon^2 h^{-4}\big) - 2C\epsilon^2 h^{-3} \kappa |p|^2 + 4C\epsilon^3 h^{-6}\, |\nabla h|^2 - 2C\epsilon^2 h^{-3} \langle \nabla h, \nabla V \rangle \\ &= |p|^2 \left(1 - 6C(E - V(q) - \tfrac{1}{2}|p|^2)^2\right) - 2C\epsilon^{1/2} \kappa |p|^2 (E - V(q) - \tfrac{1}{2}|p|^2)^{3/2} \\ &\quad + 4C(E - V(q) - \tfrac{1}{2}|p|^2)^3\, |\nabla h|^2 - 2C\epsilon^{1/2} (E - V(q) - \tfrac{1}{2}|p|^2)^{3/2} \langle \nabla h, \nabla V \rangle \\ &= |p|^2 \left(1 - 6C(E - V(q) - \tfrac{1}{2}|p|^2)^2\right) + 4C(E - V(q) - \tfrac{1}{2}|p|^2)^3\, |\nabla h|^2 \\ &\quad - 2C\epsilon^{1/2} (E - V(q) - \tfrac{1}{2}|p|^2)^{3/2} \Big[\kappa |p|^2 + \langle \nabla h, \nabla V \rangle \Big]\,. \end{aligned} \tag{3.17}$$

Now, we require $C \equiv C(E) > 0$ to satisfy

$$6C(E - \min_{\overline{\Omega}} V)^2 < 1, \tag{3.18}$$

and estimate further

$$\lambda_\epsilon(X_\epsilon)\big|_{\Sigma_\epsilon} \geq |p|^2 \left(1 - 6C(E - \min_{\overline{\Omega}} V)^2\right) + 4C\left(E - \max_{\overline{\Omega}} V - \tfrac{1}{2}|p|^2\right)^3 |\nabla h|^2 - c_\epsilon\,, \tag{3.19}$$

where

$$c_\epsilon := 2C\epsilon^{1/2} (E - \min_{\overline{\Omega}} V)^{3/2} \big[2\kappa (E - \min_{\overline{\Omega}} V) + \max_{\overline{\Omega}} |\nabla V|\big] \longrightarrow 0 \qquad \text{as } \epsilon \to 0\,. \tag{3.20}$$

For ϵ small enough, equation (3.15) and the definition of h implies that we have $|\nabla h| \geq 1/2$ in the region $\Sigma_\epsilon \cap \{|p|^2 \leq E - \max_{\overline{\Omega}} V\}$. Thus, (3.19) implies that on $\Sigma_\epsilon \cap \{|p|^2 \leq E - \max_{\overline{\Omega}} V\}$

$$\begin{aligned} \lambda_\epsilon(X_\epsilon) &\geq C(E - \max_{\overline{\Omega}} V - \tfrac{1}{2}(E - \max_{\overline{\Omega}} V))^3 - c_\epsilon \\ &= \tfrac{1}{8} C(E - \max_{\overline{\Omega}} V)^3 - c_\epsilon \,. \end{aligned} \tag{3.21}$$

On $\Sigma_\epsilon \cap \{|p|^2 \geq E - \max_{\overline{\Omega}} V\}$ we can estimate

$$\lambda_\epsilon(X_\epsilon) \geq (E - \max_{\overline{\Omega}} V)\left(1 - 6C(E - \min_{\overline{\Omega}} V)^2\right) - c_\epsilon \,. \tag{3.22}$$

Since $c_\epsilon \to 0$ as $\epsilon \to 0$, for sufficiently small ϵ we have

$$\lambda_\epsilon(X_\epsilon)\big|_{\Sigma_\epsilon} \geq \tfrac{1}{2} \min\left\{ \tfrac{1}{8} C(E - \max_{\overline{\Omega}} V)^3, (E - \max_{\overline{\Omega}} V)\left(1 - 6C(E - \min_{\overline{\Omega}} V)^2\right) \right\} \tag{3.23}$$

Hence, by setting

$$C := \frac{8}{(E - \max_{\overline{\Omega}} V)^2 + 48(E - \min_{\overline{\Omega}} V)^2} \tag{3.24}$$

we obtain

$$\lambda_\epsilon(X_\epsilon)\big|_{\Sigma_\epsilon} \geq \frac{(E - \max_{\overline{\Omega}} V)^3}{2\left[(E - \max_{\overline{\Omega}} V)^2 + 48(E - \min_{\overline{\Omega}} V)^2\right]} > 0 \tag{3.25}$$

□

Let R_ϵ be the Reeb vector field on Σ_ϵ associated to the contact form $\lambda_\epsilon|_{\Sigma_\epsilon}$. The above proposition implies that $X_\epsilon = r_\epsilon R_\epsilon$ where $r_\epsilon : \Sigma_\epsilon \to \mathbb{R}_{>0}$ is a smooth function that is bounded from below by $\Lambda(E)$. In particular, the periodic orbits of X_ϵ and R_ϵ agree up to reparamentrization. More precisely, if v is a Reeb orbit of period T then the corresponding orbit of X_ϵ has period τ_ϵ satisfying

$$\tau_\epsilon \cdot \Lambda(E) \leq T \,. \tag{3.26}$$

Since $\pi(\Sigma_\epsilon) \subset \Omega$ under the projection $\pi : T^*S^N \to S^N$ the energy hypersurface Σ_ϵ is Hamiltonianly displaceable, that is, there exists a Hamiltonian diffeomorphism $\phi_G \in \mathrm{Ham}_c(T^*S^N)$ generated by a compactly supported Hamiltonian function $G : S^1 \times T^*S^N \to \mathbb{R}$ such that

$$\phi_G(\Sigma_\epsilon) \cap \Sigma_\epsilon = \emptyset \,. \tag{3.27}$$

In fact, let $a : S^N \to \mathbb{R}$ be any function which has no critical points in $\overline{\Omega}$. If we extend a to $A := a \circ \pi : T^*S^N \to \mathbb{R}$ then the Hamiltonian flow of A displaces any compact subset of $T^*S^N|_{\overline{\Omega}}$, in particular, Σ_ϵ. Thus, if we cut off A near infinity we obtain a displacing Hamiltonian diffeomorphism in $\mathrm{Ham}_c(T^*S^N)$.

We recall that the displacement energy $e(\Sigma_\epsilon)$ is defined as

$$e(\Sigma_\epsilon) := \inf\left\{ \int_0^1 \Big[\max_{T^*S^N} G(t,\cdot) - \min_{T^*S^N} G(t,\cdot)\Big] dt \;\Big|\; \phi_G(\Sigma_\epsilon) \cap \Sigma_\epsilon = \emptyset \right\} \,. \tag{3.28}$$

Lemma 3.3. *The displacement energy of Σ_ϵ can be bounded as follows*

$$e(\Sigma_\epsilon) \leq 2(2E - 2\min_{\overline{\Omega}} V)^{1/2} \cdot \mathrm{diam}(\Omega) \tag{3.29}$$

Here $\mathrm{diam}(\Omega)$ *denotes the diameter of* $\Omega \subset \mathbb{R}^N$ *and* E *is the energy value we fixed.*

Proof. We recall that $\Sigma_\epsilon = \left\{\frac{1}{2}|p|^2 + V(q) + U_\epsilon(q) = E\right\}$. We set $R := (2E - 2\min_{\overline{\Omega}} V)^{1/2}$. Then using $U_\epsilon(q) \geq 0$ we have

$$\Sigma_\epsilon \subset \Omega \times B_R \subset \mathbb{R}^{2N} \tag{3.30}$$

where $B_R \subset \mathbb{R}^N$ is the ball around 0 of radius R. To estimate the displacement energy we choose a vector $v \in \mathbb{R}^N$ such that $(v + \Omega) \cap \Omega = \emptyset$ and set $G(q,p) := \sum v_i p_i : \mathbb{R}^{2N} \to \mathbb{R}$. Thus, the corresponding Hamiltonian diffeomorphism is $\phi_G(q,p) = (q+v,p)$. In particular, ϕ_G displaces Σ_ϵ from itself. To get a compactly supported Hamiltonian function we cut off G to zero outside an arbitrarily small neighborhood of $\Omega \times B_R$. Thus, for any $\delta > 0$ we can estimate

$$\begin{aligned} e(\Sigma_\epsilon) &\leq e(\Omega \times B_R) \\ &\leq \int_0^1 \Big[\max_{\Omega\times B_R} G(t,\cdot) - \min_{\Omega\times B_R} G(t,\cdot)\Big] dt + \delta \\ &= \max_{\Omega\times B_R} G(q,p) - \min_{\Omega\times B_R} G(q,p) + \delta \\ &\leq \max_{\Omega\times B_R} |v|\cdot|p| - \min_{\Omega\times B_R} |v|\cdot|p| + \delta \\ &= 2R|v| + \delta\ . \end{aligned} \tag{3.31}$$

Choosing the optimal vector v together with the definition of $R = (2E - 2\min_{\overline{\Omega}} V)^{1/2}$ proves the Lemma. □

The following theorem was proved by Schlenk in [Sch06], see also [CFP09, Theorem 4.9].

Theorem 3.4. *Σ_ϵ carries a Reeb orbit $v_\epsilon : \mathbb{R}/T\mathbb{Z} \to \Sigma_\epsilon$ with period T bounded by the displacement energy of Σ_ϵ, i.e.*

$$T \leq e(\Sigma_\epsilon)\ . \tag{3.32}$$

Remark 3.5. In fact, Schlenk proves a much more general existence result for closed characteristics v on displaceable hypersurfaces with bounds on the symplectic area enclosed by the closed characteristic. Since Σ_ϵ is of restricted contact type this translates into

$$T = \int_0^T v^*\lambda \leq e(\Sigma_\epsilon)\ . \tag{3.33}$$

We recall that if v_ϵ is a Reeb orbit of period T then the corresponding orbit of X_ϵ has period τ_ϵ satisfying

$$\Lambda(E)\tau_\epsilon \leq T \tag{3.34}$$

where $\Lambda(E)$ is the constant from Proposition 3.2. Combining this with Lemma 3.3 we obtain the following lemma.

Lemma 3.6. *The Hamiltonian vector field X_ϵ on Σ_ϵ has a periodic orbit of period τ_ϵ satisfying*

$$\Lambda(E)\tau_\epsilon \leq e(\Sigma_\epsilon) \leq 2(2E - 2\min_{\overline{\Omega}} V)^{1/2}\cdot \operatorname{diam}(\Omega)\ , \tag{3.35}$$

and thus

$$\tau_\epsilon \leq \frac{2(2E - 2\min_{\overline{\Omega}} V)^{1/2}\cdot \operatorname{diam}(\Omega)}{\Lambda(E)}\ . \tag{3.36}$$

This, of course, immediately implies that the Euler-Lagrangian equation corresponding to L_ϵ has a solution γ_ϵ of energy $E_\epsilon(\gamma_\epsilon) = E$ with period τ_ϵ satisfying (3.36).

For later purposes we need the additional information that the Morse index of γ_ϵ is bounded by $\frac{1}{2}\dim\Sigma_\epsilon + 1 = N+1$. It is a classical fact that under Legendre duality between $\mathscr{L}_\epsilon^E$ and K_ϵ

the Morse index and the Conley-Zehnder index agree. More precisely, if $(\Gamma,\tau) \in \mathrm{Crit}(\mathscr{L}_\epsilon^E)$ and the Reeb orbit v correspond under Legendre duality then

$$\mu_{\mathrm{Morse}}\big(\Gamma; \mathscr{L}_\epsilon^E|_{H^1\times\{\tau\}}\big) = \mu_{\mathrm{CZ}}(v) \ . \tag{3.37}$$

This identity has been proved by Viterbo in [Vit90], who extended a previous related result by Duistermaat [Dui76] (see also [LA98, Abb03] for alternative proofs).

Now, let us first assume that the functional $\mathscr{L}_\epsilon^E$ is Morse-Bott. Via Legendre duality this translates, in the Hamiltonian formulation, to the fact that on Σ_ϵ is non-degenerate, i.e. all Reeb orbits are isolated and non-degenerate, that is, the linearized Poincaré return map along a Reeb orbit has only one eigenvalue equal to 1 (which is necessarily there due to the autonomous character of the Reeb flow.) Then the proof of Theorem 4.9 in [CFP09] can be improved to show that Conley-Zehnder index of the Reeb orbit v_ϵ satisfies

$$\mu_{\mathrm{CZ}}(v_\epsilon) \in \{N, N+1\} \ . \tag{3.38}$$

In more detail, it is shown in [CFP09] that a certain moduli space would be compact if the Reeb orbit v_ϵ did not exist. This leads then to a contradiction. Assuming that Σ_ϵ is non-degenerate a closer inspection of the proof shows that a gradient flow line (in the sense of Floer) of the Rabinowitz action functional connecting the orbit v_ϵ and a maximum of an auxiliary Morse function on Σ_ϵ has to exists. Using the index formula in [CF09, Proposition 4.1] and the μ-grading for Morse-Bott homology [CF09, Appendix A] (see also the paragraph below equation (66) therein) this translates to

$$\begin{aligned} 1 &= \mu(v_\epsilon) - \mu(\max) \\ &= \mu_{\mathrm{CZ}}(v_\epsilon) + \eta(v_\epsilon) - \tfrac{1}{2} - \underbrace{\mu_{\mathrm{CZ}}(\max)}_{=0} - \tfrac{1}{2}(2N-1) \end{aligned} \tag{3.39}$$

where $\eta(v_\epsilon) \in \{0,1\}$. This summand is due to the fact that a critical point on the critical manifold represented by the periodic orbit v_ϵ has Morse index 0 or 1. The conventions for the Conley-Zehnder index in [CF09] agree with the ones here, see [CF09, Equation (60)]. Therefore, we conclude

$$\mu_{\mathrm{CZ}}(v_\epsilon) \in \{N, N+1\} \ . \tag{3.40}$$

Thus, we conclude that $\gamma_\epsilon = \pi(v_\epsilon)$ has Morse index N or $N+1$ under the assumption that $\mathscr{L}_\epsilon^E$ is Morse-Bott.

If $\mathscr{L}_\epsilon^E$ is degenerate we choose a sequence of compactly supported C^∞-small perturbations $f_n : T^*S^N \to \mathbb{R}$ such that the action functional $\mathscr{L}_\epsilon^{E,f_n}$ corresponding to the Lagrangian $L_\epsilon + f_n + E$ is Morse-Bott, we find by our previous discussion a sequence v_ϵ^n of critical points of $\mathscr{L}_\epsilon^{E,f_n}$ such that all v_ϵ^n have period uniformly bounded from above by $e(\Sigma_\epsilon) + \delta$ for some small $\delta > 0$, energy E, and Morse index N or $N+1$. Since f_n is C^∞-small and the period of v_ϵ^n is uniformly bounded (see Lemma 3.6) the sequence (v_ϵ^n) converges and thus, we obtain a critical point $\gamma_\epsilon : \mathbb{R}/T\mathbb{Z} \to \Omega$ of $\mathscr{L}_\epsilon^E$ with

$$\Lambda(E)\tau_\epsilon \le e(\Sigma_\epsilon) + \delta, \quad E_\epsilon(\gamma_\epsilon) = E, \quad \mu_{\mathrm{Morse}}(\gamma_\epsilon) \le N+1 \ . \tag{3.41}$$

Moreover, we can choose δ as small as we like. Let us summarize this discussion.

Proposition 3.7. *For any $\epsilon > 0$ and $E > \max_{\overline{\Omega}} V$ there exists a critical point $(\Gamma_\epsilon, \tau_\epsilon)$ of $\mathscr{L}_\epsilon^E$ with*

$$\begin{aligned} &\tau_\epsilon \leq \frac{2(2E - 2\min_{\overline{\Omega}} V)^{1/2} \cdot \mathrm{diam}(\Omega)}{\Lambda(E)}, \\ &E_\epsilon\big(\Gamma_\epsilon(\tfrac{t}{\tau_\epsilon})\big) = E, \\ &\mu_{\mathrm{Morse}}\big(\Gamma_\epsilon; \mathscr{L}_\epsilon^E|_{H^1\times\{\tau_\epsilon\}}\big) \leq N+1\,. \end{aligned} \tag{3.42}$$

We now have all the ingredients to prove Theorem 1.2 and Corollary 1.4.

Proof of Theorem 1.2. We fix an energy value $E > \max_{\overline{\Omega}} V$ and consider the sequence $\{(\Gamma_\epsilon, \tau_\epsilon)\}$ given in Proposition 3.7.

We first show that the sequence $\{\tau_\epsilon\}$ is uniformly bounded from below by some constant $T_1 > 0$. Indeed, assume by contradiction that $\tau_\epsilon \to 0$ up to a subsequence for $\epsilon \to 0$. Then, up to taking a further subsequence, by Proposition 2.3 we infer that Γ_ϵ converges uniformly to a constant curve $\gamma \equiv q$ with $E(\gamma) = V(q) = E$ and such that q is either a critical point of V or $q \in \partial\Omega$ and $\nabla V(q) = -a\nu(q)$ for some $a > 0$. This contradicts the assumption $E > \max_{\overline{\Omega}} V$.

Hence, we have

$$0 < T_1 \leq \tau_\epsilon \leq T_2 := \frac{2(2E - 2\min_{\overline{\Omega}} V)^{1/2} \cdot \mathrm{diam}(\Omega)}{\Lambda(E)}\,. \tag{3.43}$$

By Proposition 2.1, up to taking a further subsequence for $\epsilon \to 0$, $(\Gamma_\epsilon, \tau_\epsilon)$ converges to some (Γ, τ) in $H^1(S^1; \mathbb{R}^N) \times \mathbb{R}_{>0}$, where $T_1 \leq \tau_\epsilon \leq T_2$. Let μ be the measure given by Proposition 2.1. By Proposition 2.2 and by the uniform bound on the Morse index of Γ_ϵ, the support of μ contains at most $N+1$ points. Therefore, by Proposition 2.1, the τ-periodic curve $\gamma(t) := \Gamma(\frac{t}{\tau})$ is a τ periodic bounce orbit of the Lagrangian system given by L with energy $E(\gamma) = E$ and at most $N+1$ bounce points. □

Proof of Corollary 1.4. If the potential V is constant, say $V \equiv c$, then the solutions of the Euler-Lagrange equation of L with energy $E > c$ are straight curves with constant positive velocity, and therefore each of them will eventually bounce on $\partial\Omega$. Let us now consider the nontrivial case in which

$$\max_{\overline{\Omega}} |\nabla V| > 0, \tag{3.44}$$

and let γ be a periodic bounce orbit with energy $E(\gamma) = E > \max_{\overline{\Omega}} V$ and no bounce points. Then γ is a smooth, periodic solution of the Euler-Lagrange equation

$$\gamma'' + \nabla V(\gamma) = 0\,. \tag{3.45}$$

Since $\gamma(t) \in \overline{\Omega}$ for each $t \in \mathbb{R}$, we can estimate

$$\begin{aligned}
\mathrm{diam}(\overline{\Omega}) &\geq |\gamma(t) - \gamma(0)| \\
&= \left| \int_0^t \gamma'(s)\, ds \right| \\
&\geq |\gamma'(0)| \cdot |t| - \left| \int_0^t \int_0^s \gamma''(r)\, dr\, ds \right| \\
&\geq |\gamma'(0)| \cdot |t| - \int_0^t \int_0^s |\nabla V(\gamma(r))|\, dr\, ds \\
&= 2(E - V(\Gamma(0)))|t| - \int_0^t \int_0^s |\nabla V(\gamma(r))|\, dr\, ds \\
&\geq 2(E - \max_{\overline{\Omega}} V)^{1/2}|t| - \tfrac{1}{2}(\max_{\overline{\Omega}} |\nabla V|)t^2 \\
&\geq 2(E - \max_{\overline{\Omega}} V)^{1/2} t - \tfrac{1}{2}(\max_{\overline{\Omega}} |\nabla V|)t^2.
\end{aligned} \tag{3.46}$$

By (3.44) the above is possible only if

$$\left[2(E - \max_{\overline{\Omega}} V)^{1/2}\right]^2 - 4\mathrm{diam}(\overline{\Omega})\tfrac{1}{2} \max_{\overline{\Omega}} |\nabla V| \leq 0, \tag{3.47}$$

which can be rewritten as

$$E \leq \max_{\overline{\Omega}} V + \tfrac{1}{2}\mathrm{diam}(\overline{\Omega}) \max_{\overline{\Omega}} |\nabla V|. \tag{3.48}$$

This implies that all periodic bounce orbits with energy $E > \max_{\overline{\Omega}} V + \frac{1}{2}\mathrm{diam}(\overline{\Omega}) \max_{\overline{\Omega}} |\nabla V|$ have at least one bounce point. □

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