MARKED BOUNDARY RIGIDITY FOR SURFACES

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Abstract. We show that, on an oriented compact surface, two sufficiently C^2-close Riemannian metrics with strictly convex boundary, no conjugate points, hyperbolic trapped set for their geodesic flows, and same marked boundary distance, are isometric via a diffeomorphism that fixes the boundary. We also prove that the same conclusion holds on a compact surface for any two negatively curved Riemannian metrics with strictly convex boundary and same marked boundary distance, extending a result of Croke and Otal.

1. Introduction

In a smooth oriented compact Riemannian surface (M, g) with strictly convex boundary ∂M and no conjugate points there is a unique geodesic in each homotopy class of curves joining any given pair of points on ∂M. The *boundary distance function* can be defined as the restriction to $\partial M \times \partial M$ of the Riemannian distance function of (M, g). This function measures the length of the minimizing geodesic joining the given points on ∂M. If M is not simply connected, in general we cannot expect to recover much information on the isometry class of (M, g) from the boundary distance function alone: indeed, all the minimizing geodesics between boundary points may be contained in a rather small neighborhood of ∂M. Instead, it is natural to consider the set of lengths of all geodesics joining boundary points, and to study the associated rigidity problem. More precisely, we consider the *lens data* of g, which consists of a pair of functions (ℓ_g, S_g) defined as follows. The *scattering map* S_g takes a vector of unit length $(x, v) \in T_{\partial M}M$ pointing inside M, and returns the exit tangent vector $S_g(x, v) = (x', v') \in T_{\partial M}M$ obtained by following the geodesic $\alpha_{x,v}$ of (M, g) starting at x with tangent vector v. The *length function* ℓ_g takes an inward-pointing unit vector $(x, v) \in T_{\partial M}M$ and returns the length of the geodesic $\alpha_{x,v}$; note that $\ell_g(x, v)$ may be infinite at some points (x, v). The lens rigidity problem asks whether the lens data (ℓ_g, S_g) determines the metric g up to an isometry that is the identity on ∂M. In this paper, we also consider another related quantity that we call the *marked boundary distance*, defined as follows. There is a bundle $\mathcal{P}_{\partial M} \to \partial M \times \partial M$ whose fiber over $(x, x') \in \partial M \times \partial M$ is the set of homotopy classes of curves joining x and x'. The *marked boundary distance* is the function on $\mathcal{P}_{\partial M}$ given by

$$(x, x'; [\gamma]) \mapsto \text{length}(\alpha(x, x'; [\gamma])),$$

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where $\alpha(x,x';[\gamma])$ is the unique geodesic with endpoints x and x' in the homotopy class given by $[\gamma]$. The question that we address here is the rigidity of Riemannian metrics with the same marked boundary distance, much in the spirit of Otal's paper [Ota90a] on the marked length spectrum rigidity for closed negatively-curved surfaces. We remark that the marked boundary distance function gives the same information as the lens data on the universal cover of M, see Section 2.

In order to describe our results, we first need to recall a few notions. As usual, we denote by $SM = \{(x,v) \in TM \mid g_x(v,v)=1\}$ the unit tangent bundle, and by φ_t the geodesic flow on SM at time t, which is generated by the geodesic vector field X. For all points $(x,v) \in SM$, the geodesic with initial point x and tangent vector v has either infinite length or exits M at some boundary point $x' \in \partial M$. We define the *trapped set* $K \subset SM^\circ$ of g to be the set of points that do not reach the boundary in forward nor in backward time:

$$K := \big\{(x,v) \in SM^\circ \bigm| \forall t \in \mathbb{R},\ \varphi_t(x,v) \in SM^\circ\big\}.$$

Due to the strict convexity of ∂M, K is compact and invariant by the geodesic flow. In Section 2, we will prove that if a Riemannian metric g on a compact surface has geodesic flow with hyperbolic trapped set (see the definition before equation (2.3)), strictly convex boundary, and no conjugate points, then the same is true for all smooth Riemannian metrics in a sufficiently small C^2-neighborhood of g.

Our first theorem is a local marked boundary rigidity result for surfaces with hyperbolic trapped set and no conjugate points. Its proof is based on the recent work [Gui14] of the first author.

Theorem 1. *Let (M,g_1) be a smooth oriented compact Riemannian surface with strictly convex boundary, no conjugate points, and whose trapped set K is hyperbolic. There exists $\delta > 0$ such that, for each smooth Riemannian metric g_2 on M satisfying $\|g_2 - g_1\|_{C^2} < \delta$ and having the same marked boundary distance of g_1, there is a smooth diffeomorphism $\phi : M \to M$ with $\phi^* g_2 = g_1$ and $\phi|_{\partial M} = \mathrm{Id}$.*

Our second theorem deals with the case where one of the metric has negative curvature, and can be seen as an extension of a celebrated result due to Croke [Cro90]and Otal [Ota90b]. The method that we employ is essentially the one of [Ota90b].

Theorem 2. *Let M be a smooth oriented compact surface with boundary. Let g_1 be a smooth Riemannian metric on M with negative curvature and strictly convex boundary, and g_2 a smooth Riemannian metric on M with strictly convex boundary, no conjugate points and trapped set of zero Liouville measure. If g_1 and g_2 have the same marked boundary distance, there is a diffeomorphism $\psi : M \to M$ with $\psi^* g_2 = g_1$ and $\psi|_{\partial M} = \mathrm{Id}$.*

We remark that the assumptions on the Riemannian metric g_2 in Theorem 2 are satisfied if (M,g_2) has negative curvature and strictly convex boundary.

In order to put our theorems into perspective, let us recall a few results on related rigidity questions for surfaces. Riemannian metrics with strictly convex boundary, no conjugate points, and empty trapped set are called *simple metrics*. Their underlying compact surfaces are topological disks. In 1981, Michel conjectured that simple Riemannian metrics on compact surfaces are boundary rigid [Mic82], i.e. the boundary distance function determines the Riemannian metric within the class of simple metrics up to isometry. Mukhometov [Muk81] proved this conjecture under the extra assumption that the simple metrics considered belong to a given conformal class (see also Croke [Cro91] for a simpler proof). Later, Croke [Cro90] and Otal [Ota90b] proved Michel's conjecture for negatively-curved simple metrics. This was extended to non-positively curved simple metrics by Croke, Fathi, and Feldman [CFF92]. The conjecture for simple surfaces was finally established by Pestov and Uhlmann [PU05]. As for non simple metrics on compact surfaces, Croke and Herreros [CH16] proved that a negatively-curved cylinder, the flat cylinder, and the flat Möbius strip are lens rigid. In [Gui14], the first author showed that the scattering map S_g determines the compact manifold and the Riemannian metric, within the class considered in Theorem 1, up to a conformal diffeomorphism.

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2. Manifolds with hyperbolic trapped set and no conjugate points

2.1. **Geometric and dynamical preliminaries.** Let (M, g) be a compact Riemannian surface with strictly convex boundary. As before, we denote by X its geodesic vector field on SM, and by φ_t the corresponding geodesic flow. We also equip SM with a contact form λ, which is obtained by pulling back the canonical Liouville form on T^*M via the diffeomorphism $(x, v) \mapsto (x, g_x(v, \cdot))$. The Liouville measure μ_g on SM is defined as the measure associated to the density $|\lambda \wedge d\lambda|$. We consider the pull-back Riemannian metric $\widetilde{g} = \pi^* g$ on the universal cover $\pi : \widetilde{M} \to M$. The fundamental group $\pi_1(M) = \pi_1(M, x_0)$, for some fixed $x_0 \in M$, acts by isometries on $(\widetilde{M}, \widetilde{g})$. The universal cover $\widetilde{M}$ is non-compact and has non-compact boundary $\partial\widetilde{M}$ that is a countable union of connected components and projects to ∂M. The unit tangent bundle $S\widetilde{M}$ of $\widetilde{M}$ is also a cover of SM and, by abuse of notation, we still denote by $\pi : S\widetilde{M} \to SM$ the covering map. We will denote by $\widetilde{\varphi}_t$ and $\mu_{\widetilde{g}}$ the geodesic flow and the Liouville measure on $S\widetilde{M}$. Notice that $\widetilde{\varphi}_t$ is a lift of φ_t, i.e. $\varphi_t \circ \pi = \pi \circ \widetilde{\varphi}_t$, and if $\gamma.(x, v)$ denotes the natural action of $\pi_1(M)$ on SM, we have

$$\widetilde{\varphi}_t(\gamma \cdot (x, v)) = \gamma \cdot (\widetilde{\varphi}_t(x, v)), \qquad \forall (x, v) \in S\widetilde{M},\ \gamma \in \pi_1(M).$$

We define the incoming $(-)$, outgoing $(+)$ and tangent (0) boundaries of SM

$$\partial_\mp SM := \{(x, v) \in \partial SM \mid \pm \langle v, \nu \rangle > 0\},$$
$$\partial_0 SM := \{(x, v) \in \partial SM \mid \langle v, \nu \rangle = 0\}.$$

where ν is the inward-pointing unit normal vector field to ∂M. For each $(x, v) \in SM$, we define the time of escape from M by

$$\ell_g(x, v) := \sup \{t \geq 0 \mid \varphi_s(x, v) \in SM^\circ \ \ \forall s < t\} \subset [0, +\infty].$$

We define $\partial_\pm\widetilde{M}$, $\partial_0 S\widetilde{M}$, and the time of escape $\ell_{\widetilde{g}} : S\widetilde{M} \to [0,\infty]$ analogously. Notice that $\ell_{\widetilde{g}} = \ell_g \circ \pi$. We define the incoming ($-$) and outgoing ($+$) tails in SM as

$$\Gamma_\mp := \{(x,v) \in SM \mid \ell_g(x,\pm v) = \infty\},$$

and the trapped set for the geodesic flow as

$$K := \Gamma_+ \cap \Gamma_-. \tag{2.1}$$

Using the strict convexity of ∂M, it is straightforward to see that that $\Gamma_\pm$ and K are compact, and K is a subset of SM° invariant by the geodesic flow. By the strict convexity of the boundary, the argument in [DG14a, Section 5.1] shows that

$$\mu_g(K) = 0 \iff \mu_g(\Gamma_+ \cup \Gamma_-) = 0. \tag{2.2}$$

We say that K is *hyperbolic* when, for all $y \in K$, there is a $d\varphi_t$-invariant splitting

$$T_y(SM) = \mathbb{R}X(y) \oplus E_u(y) \oplus E_s(y), \tag{2.3}$$

where E_u and E_s are continuous subbundles over K satisfying, for some $C, \alpha > 0$ and for all $y \in K$,

$$\begin{aligned} \zeta \in E_s(y) \qquad &\text{if and only if} \qquad \|d\varphi_t(y)\zeta\|_g \le Ce^{-\alpha t}\|\zeta\|_g, \quad \forall t \ge 0,\\ \zeta \in E_u(y) \qquad &\text{if and only if} \qquad \|d\varphi_t(y)\zeta\|_g \le Ce^{\alpha t}\|\zeta\|_g, \quad \forall t \le 0. \end{aligned} \tag{2.4}$$

The norms in these expressions are given by the Sasaki metric on SM, see e.g. [Pat99]. We define the vertical bundle by

$$V := \ker(d\pi) \subset T(SM),$$

where $\pi : SM \to M$ is the base projection. We say that $x_-, x_+ \in M$ are *conjugate points* with respect to the geodesic flow φ_t if there exists $t_0 \in \mathbb{R}$ such that $\varphi_{t_0}(x_-) = x_+$ and $(d\varphi_{t_0}(x_-)V(x_-)) \cap V(x_+) \neq \{0\}$. By [DG14b, Lemma 2.10], there is a continuous extension $E_- \subset T_{\Gamma_-}(SM)$ of E_s over Γ_- that is invariant by the geodesic flow and satisfies contraction estimates of the form (2.4). If (M,g) has no conjugate points, $V \cap E_- = \{0\}$, see [Gui14, Section 2.2]. There is an analogous extension $E_+ \subset T_{\Gamma_+}(SM)$ of E_u over Γ_+.

Proposition 2.1. *Let (M,g) be a compact Riemannian surface with strictly convex boundary, no conjugate points, and with hyperbolic trapped set for its geodesic flow. There exists $\delta > 0$ such that, for each Riemannian metric g' satisfying $\|g'-g\|_{C^2} < \delta$, the Riemannian surface (M,g') has strictly convex boundary, no conjugate points, and hyperbolic trapped set for its geodesic flow.*

Proof. We denote by φ_t the geodesic flow of the Riemannian metric g, by $\Gamma_\mp \subset SM$ its incoming and outgoing tails, and by $K \subset SM^\circ$ its trapped set as usual. Consider the family of open sets

$$U_{\tau,\pm} := \big\{ y \in SM \ \big|\ \varphi_{\mp t}(y) \in SM^\circ \ \ \forall t \in [0,\tau] \big\}, \tag{2.5}$$

for $\tau > 0$. This family forms a fundamental system of open neighborhoods of $\Gamma_\pm$, that is, for every open neighborhood $U \subset SM$ of $\Gamma_\pm$ there exists $\tau > 0$ large enough such that $U_{\tau,\pm} \subset U$. Analogously, the intersections $U_\tau := U_{\tau,+} \cap U_{\tau,-}$ form a fundamental system of open neighborhoods of the trapped set K. We refer the reader to [DG14b] for a proof of these facts.

Consider the hyperbolic splitting (2.3) over K, and the extensions $E_\mp \subset T_{\Gamma_\mp}(SM)$ of the stable and unstable bundles. We further extend $E_\mp$ continuously over a neighborhood $U_{\tau_0,\mp}$ of $\Gamma_\pm$, and define the stable and unstable cones as

$$C_{\mp,\rho}(y) := \big\{\xi+\eta \;\big|\; \xi \in E_\mp(y),\ \eta \in E_\pm(y)\oplus \mathbb{R}X(y),\ \|\eta\|_g \le \rho\|\xi\|_g\big\},$$

where $y \in U_{\tau_0,\mp}$ and $\rho > 0$. We claim that, for all large enough real numbers $\tau_0 > 0$ and $\alpha_0 > 0$, we have

$$d\varphi_{\pm t}(y)V(y) \Subset C_{\pm,\alpha_0}(\varphi_{\pm t}(y)), \qquad \forall y \in U_{2\tau_0,\mp},\ \forall t \in [\tau_0, 2\tau_0]. \tag{2.6}$$

Indeed, if this is not the case, by a compactness argument, for all sequences $\tau_n \to +\infty$ there exist $y_n^\mp \in U_{2\tau_n,\mp}$ and $t_n > \tau_n$ such that $d\varphi_{\pm t_n}(y_n^\mp)V(y_n^\mp) \cap E_\mp(\varphi_{\pm t_n}(y_n^\mp)) \neq 0$. Since the Riemannian manifold (M, g) has no conjugate points, the vertical bundle V is contained in a conic neighborhood $C_{\pm,\rho}$, for some $\rho > 0$, whose closure does not contain $E_\mp$. Therefore, we can apply Lemma 2.11 of [DG14b] to deduce that the distance of $d\varphi_{\pm t_n}(y_n^\mp)V(y_n^\mp)$ to E_+ in the Grassmannian bundle of SM tends to 0 as $n \to \infty$, which gives a contradiction.

We fix $\tau_0 > 0$ large enough so that (2.6) holds and, for some $\rho_0 > 0$ small enough, we have

$$C_{+,\rho_0}(y) \cap C_{-,\rho_0}(y) = \{0\}, \qquad \forall y \in U_{\tau_0}.$$

By the hyperbolicity of the flow on K, for each pair of positive numbers $\alpha \ge \rho > 0$ there exists $\overline{\tau} = \overline{\tau}(\alpha, \rho) > 2\tau_0$ large enough such that, for all $t \ge \overline{\tau}$ and $y \in U_{\tau_0} \cap \varphi_{\mp t}^{-1}(U_{\tau_0})$,

$$\begin{aligned} &d\varphi_{\mp t}(y)C_{\mp,\alpha}(y) \Subset C_{\mp,\rho}(\varphi_{\mp t}(y)),\\ &\|d\varphi_{\mp t}(y)\zeta\|_g \ge 4\|\zeta\|_g, \quad \forall \zeta \in C_{\mp,2\rho}(y). \end{aligned} \tag{2.7}$$

There exists $\delta = \delta(\overline{\tau}) > 0$ such that, for any Riemannian metric g' on M satisfying $\|g' - g\|_{C^2} < \delta$, the boundary of (M, g') is strictly convex and no pair of points of (M, g') are conjugate along a geodesic of length less than or equal to $4\overline{\tau}$. Let $\psi : SM \to S'M$ be the diffeomorphism given by $\psi(x, v) = (x, v/\|v\|_{g'})$, where $S'M$ is the unit tangent bundle of M with respect to g'. Since ψ preserves the fibers of the unit tangent bundles, its differential maps the vertical subbundle $V \subset T(SM)$ to the vertical subbundle of $T(S'M)$. From now on, we identify SM with $S'M$ by means of ψ, and thus we see the geodesic flow φ'_t of g' as a flow on SM. We define the open sets $U'_{\tau,\pm}$ analogously to (2.5), that is,

$$U'_{\tau,\pm} := \big\{y \in SM \;\big|\; \varphi'_{\mp t}(y) \in SM^\circ \ \ \forall t \in [0, \tau]\big\},$$

and $U'_\tau := U'_{\tau,-} \cap U'_{\tau,+}$. Notice that φ'_t is $\mathcal{O}(\delta)$-close to φ_t in the C^1-topology for all $t \in [0, 2\tau_0]$. In particular, up to further reducing $\delta > 0$, we have $U'_{2\tau_0,\mp} \subset U_{\tau_0,\mp}$, and the following holds: by (2.7), for all $t \in [\overline{\tau}, 2\overline{\tau}]$ and $y \in U'_{2\tau_0} \cap \varphi'_{\pm t}(U'_{2\tau_0})$ we have

$$\begin{aligned} &d\varphi'_{\mp t}(y)C_{\mp,\alpha_0}(y) \Subset C_{\mp,\rho_0}(\varphi'_{\mp t}(y)),\\ &\|d\varphi'_{\mp t}(y)\zeta\|_{g'} \ge 2\|\zeta\|_{g'}, \quad \forall \zeta \in C_{\mp,2\rho_0}(y); \end{aligned} \tag{2.8}$$

in addition, by (2.6), we have

$$d\varphi'_{\pm t}(y)V(y) \Subset C_{+,\alpha_0}(\varphi'_{\pm t}(y)), \qquad \forall y \in U'_{2\tau_0,\mp},\ t \in [\tau_0, 2\tau_0]. \tag{2.9}$$

Equations (2.8) and (2.9) imply that

$$d\varphi'_{\pm t}(y)V(y) \Subset C_{\pm,\rho_0}(\varphi'_{\pm t}(y)), \qquad \forall y \in U'_{2\tau_0,\mp} \cap \varphi'_{\mp t}(U'_{2\tau_0}),\ t \geq 2\overline{\tau}, \tag{2.10}$$

By [KH95, Proposition 17.4.4], the set

$$K' := \bigcap_{t\in\mathbb{R}} \varphi'_t(U'_{2\tau_0}),$$

which is the trapped set for the geodesic flow φ'_t, is hyperbolic. We claim that the Riemannian metric g' has no conjugate points. Indeed, assume that two points $\gamma(0)$ and $\gamma(4\tau)$ are conjugate along a unit-speed geodesic $\gamma : [0, 4\tau] \to M$ of (M, g'). By the property of the Riemannian metric g', we must have $\tau > \overline{\tau}$. We set

$$y_- := (\gamma(0), \dot\gamma(0)), \qquad y_+ := (\gamma(4\tau), \dot\gamma(4\tau)).$$

Notice that $y_- \in U'_{2\tau_0,-}$ and $y_+ \in U'_{2\tau_0,+}$. Therefore, by (2.10), we have

$$d\varphi'_{\pm 2\tau}(y_\mp)V(y_\mp) \Subset C_{\pm,\rho_0}(\varphi'_{\pm 2\tau}(y_\mp)).$$

Since $C_{+,\rho_0} \cap C_{-,\rho_0} = \{0\}$ over $U'_{2\tau_0}$, we deduce that $d\varphi'_{4\tau}(y_-)V(y_-) \cap V(y_+) = \{0\}$, which contradicts the facts that $\gamma(0)$ and $\gamma(4\tau)$ are conjugate along γ. □

We now recall Santaló's formula corresponding to the disintegration of the Liouville measure μ_g on geodesics. Let $h := g|_{T\partial M}$ be the metric induced on the boundary. There is a natural measure on ∂SM, defined by

$$d\mu_{g,\nu}(x, v) := |g_x(v, \nu_x)|\, d\mathrm{vol}_h(x)\, dS_x(v). \tag{2.11}$$

where ν is the inward-pointing unit normal vector field to ∂M, $d\mathrm{vol}_h$ is the Riemannian volume form on $(\partial M, h)$, and dS_x is the volume measure on the fiber S_xM induced by g. Assume that $\mu_g(\Gamma_- \cup \Gamma_+) = 0$, or equivalently $\mu_g(K) = 0$ according to (2.2). Santaló's formula reads

$$\int_{SM} f d\mu_g = \int_{\partial_- SM\setminus\Gamma_-} \int_0^{\ell_g(x,v)} f(\varphi_t(x, v))dt\, d\mu_{g,\nu}(x, v), \qquad \forall f \in L^1(SM). \tag{2.12}$$

In particular, for $f \equiv 1$, Santaló's formula computes the Riemannian volume of SM as

$$\mathrm{Vol}_g(SM) = \int_{\partial_- SM\setminus\Gamma_-} \ell_g(x, v)\, d\mu_{g,\nu}(x, v). \tag{2.13}$$

We define the incoming and outgoing tails in $S\widetilde{M}$ as before by

$$\widetilde\Gamma_\mp := \{(x, v) \in SM \mid \ell_{\widetilde g}(x, \pm v) = \infty\}.$$

Notice that $\pi(\widetilde\Gamma_\mp) = \Gamma_\mp$. If K is hyperbolic, the sets $\Gamma_\pm$ have measure zero with respect to μ_g, see [Gui14, Section 2.4]; in particular, $\widetilde\Gamma_\pm$ have measure zero with respect to $\mu_{\widetilde g}$ as well. We denote by $d\mu_{\widetilde{g},\nu}$ the lift of $d\mu_{g,\nu}$ to $\partial S\widetilde{M}$. Since $\widetilde{\Gamma}_\pm$ have also measure zero with respect to $\mu_{\widetilde{g}}$, Santaló's formula on $S\widetilde{M}$ reads

$$\int_{S\widetilde{M}} f d\mu_{\widetilde{g}} = \int_{\partial_- S\widetilde{M}\setminus\widetilde{\Gamma}_-} \int_0^{\ell_{\widetilde{g}}(x,v)} f(\widetilde{\varphi}_t(x,v))dt\, d\mu_{\widetilde{g},\nu}(x,v), \qquad \forall f \in L^1(S\widetilde{M}). \tag{2.14}$$

2.2. **Scattering map, lens data, and boundary distance functions.** We define the *scattering map* of (M,g) as follows

$$S_g : \partial_- SM \setminus \Gamma_- \to \partial_+ SM \setminus \Gamma_+, \quad S_g(x,v) := \varphi_{\ell_g(x,v)}(x,v). \tag{2.15}$$

The *lens data* of (M,g) is the pair of functions $(S_g, \ell_g|_{\partial_- SM})$, which amounts to knowing the length of all geodesics joining boundary points and their tangent vectors at the boundary. The analogous definitions hold for the universal cover $(\widetilde{M},\widetilde{g})$. In particular, the scattering map

$$S_{\widetilde{g}} : \partial_- S\widetilde{M} \setminus \widetilde{\Gamma}_- \to \partial_+ S\widetilde{M} \setminus \widetilde{\Gamma}_+, \quad S_{\widetilde{g}}(x,v) := \widetilde{\varphi}_{\ell_{\widetilde{g}}(x,v)}(x,v) \tag{2.16}$$

satisfies $S_g \circ \pi = \pi \circ S_{\widetilde{g}}$ and is $\pi_1(M)$-equivariant, i.e.

$$S_{\widetilde{g}}(\gamma\cdot(x,v)) = \gamma\cdot(S_{\widetilde{g}}(x,v)), \qquad \forall (x,v)\in \partial_- S\widetilde{M}\setminus\widetilde{\Gamma}_-,\ \gamma\in\pi_1(M). \tag{2.17}$$

The lens data of $(\widetilde{M},\widetilde{g})$ is the pair $(S_{\widetilde{g}}, \ell_{\widetilde{g}}|_{\partial_- S\widetilde{M}})$, where $\ell_{\widetilde{g}} = \ell_g\circ\pi$.

The following result is well known to the experts.

Lemma 2.2. *Let (M,g) be a smooth manifold with smooth, strictly convex boundary and no conjugate points. Let α be a geodesic with endpoints $x,x'\in M$. If γ is any other smooth curve in M with endpoints x,x' that is homotopic to α with a homotopy fixing the endpoints, then the length of γ is larger than the length of α.*

Proof. Let $\Pi := \{\gamma \in W^{1,2}([0,1];M) \mid \gamma(0)=x, \gamma(1)=x'\}$, where $W^{1,2}([0,1];M)$ is the Hilbert manifold of absolutely continuous maps $\gamma:[0,1]\to M$ whose energy

$$E(\gamma) := \int_0^1 g_{\gamma(t)}(\dot\gamma(t),\dot\gamma(t))\,dt$$

is finite. We also consider the finite dimensional reduction of this space, that is, the space Π_k of all curves $\gamma\in\Pi$ such that, for all $i=0,...,k-1$, the curve $\gamma|_{[i/k,(i+1)/k]}$ is a geodesic of length smaller than $\mathrm{injrad}(M,g)$. For all $a>0$ there exists $k\in\mathbb{N}$ large enough so that the inclusion

$$\Pi_k \cap E^{-1}(-\infty,a) \hookrightarrow E^{-1}(-\infty,a) \tag{2.18}$$

is a homotopy equivalence. We refer the reader to Milnor [Mil63, Section 16] for this and other properties of the space Π_k. The connected components of Π are homotopy classes of curves in M with fixed endpoints x and x'. We choose a connected component $\Pi'\subset\Pi$ containing a geodesic α. First, we notice that a curve α' which minimizes the energy E in Π' is in fact a smooth geodesic that intersects ∂M only at its endpoints x,x'. This follows from the strict convexity of the boundary (see for instance the arguments given in [Ota90b, Section 1]). In particular, the minimizers of $E|_{\Pi'}$ lie in the interior of Π'. The critical points of the energy $E|_{\Pi'}$ are the geodesics in the connected component Π'. Since

g has no conjugate points, the Morse index Theorem implies that the Hessian of E at the critical points is positive definite (see for instance Milnor [Mil63, Section 15]). The same is true for the restriction of E to Π_k. We will use a mountain pass argument to prove that α is the unique minimizer in Π'. If α' is another local minimizer $E|_{\Pi'}$, consider a continuous path $A : [0,1] \to \Pi$ joining α and α'. We set $a > \max\{E(A(r)) \mid r \in [0,1]\}$, and choose $k \in \mathbb{N}$ large enough so that the inclusion (2.18) is a homotopy equivalence. In particular α and α' belong to the same connected component of $\Pi_k \cap E^{-1}(-\infty, a)$. We denote by E_k the restriction of E to $\Pi_k \cap E^{-1}(-\infty, a)$, and by $\mathcal{P}$ the space of continuous paths $B : [0,1] \to \Pi_k \cap E^{-1}(-\infty, a)$ with $B(0) = \alpha$ and $B(1) = \alpha'$. Since the boundary of M is strictly convex, a well known computation (see [GKM75, page 252]) shows that the flow of $-\nabla E_k$ is positively complete. Therefore, the minimax value

$$b := \inf_{B\in\mathcal{P}} \max_{s\in[0,1]} E_k(B(s))$$

is a critical value of E_k. By a theorem of Hofer [Hof85], there exists a critical point α'' with $E_k(\alpha'') = b$ that is not a local minimum of E_k. By the Morse index Theorem, α'' must contain a pair of conjugate points. This gives a contradiction. □

We denote by $\mathcal{P} \to M \times M$ the bundle whose fiber $\mathcal{P}_{(x,x')}$ over $(x, x') \in M \times M$ is the set homotopy classes of curves $\gamma : [0,1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = x'$. The fundamental group $\pi_1(M, x)$ acts freely and transitively on every fiber by concatenation: given a homotopy class $[\gamma] \in \mathcal{P}_{(x,x')}$ and an element $[\zeta] \in \pi_1(M, x)$, the action is given by $[\zeta] \cdot [\gamma] = [\zeta * \gamma]$. Therefore, the $\mathcal{P}_{(x,x')}$ is in one-to-one correspondence with $\pi_1(M, x)$. We define the *marked distance* function

$$d_g : \mathcal{P} \to [0, \infty), \quad d_g(x, x'; [\gamma]) = \text{length}(\alpha_g(x, x'; [\gamma])),$$

where $\alpha_g(x, x'; [\gamma])$ is the unique geodesic with endpoints x and x' in the homotopy class $[\gamma] \in \mathcal{P}_{(x,x')}$. We denote by $\mathcal{P}_{\partial M}$ the restriction of the bundle $\mathcal{P}$ to $\partial M \times \partial M$.

A consequence of Lemma 2.2 is that for each pair of points x, x' on $\widetilde{M}$, there is a unique geodesic $\alpha_{\widetilde{g}}(x, x')$ joining x and x' in $\widetilde{M}$. We define the distance function

$$d_{\widetilde{g}} : \widetilde{M} \times \widetilde{M} \to [0, \infty), \qquad d_{\widetilde{g}}(x, x') = \text{length}(\alpha_{\widetilde{g}}(x, x')). \tag{2.19}$$

Notice that

$$d_{\widetilde{g}}(x, x') = d_g(\pi(x), \pi(x'); [\pi \circ \alpha_{\widetilde{g}}(x, x')]).$$

We call *marked boundary distance* the restriction $d_g|_{\mathcal{P}_{\partial M}}$, the knowledge of which is equivalent to the knowledge of $d_{\widetilde{g}}|_{\partial\widetilde{M}\times\partial\widetilde{M}}$.

The next two lemmas are well known for simple metrics. Since our assumptions are weaker, we provide the proofs for the reader's convenience.

Lemma 2.3. *Let M be a compact manifold with smooth boundary, and g_1, g_2 two Riemannian metrics on M with the same boundary distance function (i.e. $d_{g_1} = d_{g_2}$ on $\partial M \times \partial M$), and such that the boundary ∂M is convex with respect to both metrics. Then there exists a diffeomorphism $\psi : M \to M$ that is the identity on ∂M and such that g_2 and $\psi^* g_1$ coincide at all points of ∂M.*

Proof. Let $\iota : \partial M \hookrightarrow M$ be the inclusion map of the boundary inside our compact manifold. We set $h_1 := \iota^* g_1$ and $h_2 := \iota^* g_2$. We claim that these two Riemannian metrics coincide. Indeed, since the boundary ∂M is convex for both g_1 and g_2, their Riemannian distances are smooth in a neighborhood of the diagonal submanifold of $M \times M$. For any given $(x, v) \in T\partial M$, if $\gamma : [0, 1] \to \partial M$ is any smooth curve such that $\gamma(0) = x$ and $\dot\gamma(0) = v$, we have

$$h_1(v, v)^{1/2} = \lim_{t\to 0} \frac{d_{g_1}(\gamma(0), \gamma(t))}{t} = \lim_{t\to 0} \frac{d_{g_2}(\gamma(0), \gamma(t))}{t} = h_2(v, v)^{1/2}.$$

Now, for each $x \in \partial M$, we denote by $\nu_1(x), \nu_2(x) \in T_x M$ the inward-pointing unit normal tangent vectors with respect to g_1 and g_2 respectively. For $i = 1, 2$, we introduce the map

$$\phi_i : \partial M \times [0, \epsilon_i) \to M, \qquad \phi_i(x, t) = \exp_x(t\nu_i(x)),$$

which, up to choosing $\epsilon_i > 0$ small enough, is a well defined diffeomorphism onto a collar neighborhood of the boundary ∂M. For $\delta_0 > 0$ small enough, the map

$$\theta : \partial M \times [0, \delta_0) \to \partial M \times [0, \epsilon_1), \qquad \theta(x, t) = \phi_1^{-1} \circ \phi_2(x, t)$$

is a well defined diffeomorphism onto its image. We write this diffeomorphism as $\theta(x, t) = (\theta_1(x, t), \theta_2(x, t))$, where $\theta_1(x, t) \in \partial M$ and $\theta_2(x, t) \in [0, \epsilon_1)$. Up to reducing $\delta_0 > 0$, we have that $\frac{\partial}{\partial t}\theta_2$ is everywhere positive, and for all $t \in [0, \delta_0)$ the map $x \mapsto \theta_1(x, t)$ is a diffeomorphism onto its image. For $\delta_1 \in [0, \delta_0/2)$ sufficiently small, we can find a smooth function $\theta_2' : \partial M \times [0, \delta_0) \to [0, \delta_0)$ such that $\frac{\partial}{\partial t}\theta_2'$ is everywhere positive, $\theta_2'(\cdot, t) = \theta_2(\cdot, t)$ for all $t \in [0, \delta_1]$, and $\theta_2'(\cdot, t) \equiv t$ for all $t \in [\delta_0/2, \delta_0)$. Let $\delta_2 \in (0, \delta_1/4)$ be a constant that we will fix sufficiently small in a moment. We choose a smooth function $\chi : [0, \delta_0) \to [0, \delta_0)$ such that $\chi(t) = t$ for all $t \in [0, \delta_2]$, $\chi(t) = 0$ for all $t \in [3\delta_2, \delta_0]$, and $|\dot\chi(t)| \leq 1$ for all $t \in [0, \delta_0)$. We define the map

$$\theta_1' : \partial M \times [0, \delta_0) \to \partial M, \qquad \theta_1'(x, t) = \theta_1(x, \chi(t)).$$

Notice that, for all $t \in [0, \delta_0)$, the map $x \mapsto \theta_1'(x, t)$ is a diffeomorphism onto its image, and such diffeomorphism is the identity if $t = 0$ or if $t \in [3\delta_2, \delta_0]$. We define

$$\theta' : \partial M \times [0, \delta_0) \to \partial M \times [0, \delta_0), \qquad \theta'(x, t) = (\theta_1'(x, t), \theta_2'(x, t)).$$

This map is clearly bijective and, up to choosing $\delta_2 > 0$ sufficiently small, its differential is everywhere bijective. Therefore, θ' is a diffeomorphism that coincides with θ on $\partial M \times [0, \delta_2)$, and with the identity on $\partial M \times (\delta_0/2, \delta_0)$. Finally, we set

$$\phi_1' : \partial M \times [0, \delta_0) \to M, \qquad \phi_1'(x, t) = \phi_1 \circ \theta'(x, t).$$

Notice that ϕ_1' is a diffeomorphism onto its image, coincides with ϕ_2 on $\partial M \times [0, \delta_2)$, and coincides with ϕ_1 on $\partial M \times (\delta_0/2, \delta_0)$. We set $U := \phi_1'(\partial M \times [0, \delta))$ and $\psi := \phi_1 \circ (\phi_1')^{-1} : U \to M$. Since ψ is equal to the identity outside a neighborhood of ∂M, we can extend it to a diffeomorphism $\psi : M \to M$ by setting $\psi(x) = x$ for all $x \notin U$.

We claim that ψ^*g_1 coincides with g_2 on all points $x \in \partial M$. Indeed, ψ fixes the boundary ∂M, and therefore

$$\iota^*\psi^*g_1 = \iota^*g_1 = \iota^*g_2. \tag{2.20}$$

Moreover ψ coincides with $\phi_1 \circ \phi_2^{-1}$ on a neighborhood of ∂M. Fix a point $x \in \partial M$, and consider the unit-speed g_2-geodesic $\gamma_2 : [0, \epsilon) \to U$ such that $\gamma_2(0) = x$ and $\dot\gamma_2(0) = \nu_2(x)$. We set $\gamma_1 := \psi \circ \gamma_2$. Notice that, if $\epsilon \in (0, \delta')$, we have $\gamma_1(t) = \phi_1(x, t)$ and $\gamma_2(t) = \phi_2(x, t)$. Therefore the curve γ_1 is the unit-speed g_1-geodesic such that $\gamma_1(0) = x$ and $\dot\gamma_1(0) = \nu_1(x)$, and in particular $d\psi(x)\nu_2(x) = \nu_1(x)$. This implies that, for all $v \in T_x(\partial M)$, we have

$$\psi^*g_1(\nu_2(x), v) = g_1(d\psi(x)\nu_2(x), d\psi(x)v) = g_1(\nu_1(x), v) = 0,$$

which, together with (2.20), completes the proof. □

Lemma 2.4. *Let M be a compact manifold with boundary, and g_1, g_2 two Riemannian metrics on M with no conjugate points and such that the boundary ∂M is strictly convex with respect to both metrics. Assume that g_1 and g_2 coincide at all points of ∂M. Then, they have the same marked boundary distance function, i.e. $d_{\widetilde g_1} = d_{\widetilde g_2}$ on $\partial\widetilde M \times \partial\widetilde M$, if and only if their lens data on $\widetilde M$ agree, i.e. $(S_{\widetilde g_1}, \ell_{\widetilde g_1}|_{\partial_- S\widetilde M}) = (S_{\widetilde g_2}, \ell_{\widetilde g_2}|_{\partial_- S\widetilde M})$.*

Proof. Since g_1 and g_2 are metrics without conjugate points and the boundary ∂M is strictly convex for both of them, any pair of distinct points $x, x' \in \partial\widetilde M$ is joined by a unique $\widetilde g_1$-geodesic $\alpha_{\widetilde g_1}(x, x')$ and by a unique $\widetilde g_2$-geodesic $\alpha_{\widetilde g_2}(x, x')$. For all distinct points $x, x' \in \partial\widetilde M$ we have $d_{\widetilde g_1}(x, x') = \ell_{\widetilde g_1}(x, v)$, where v is the unit tangent vector to $\alpha_{\widetilde g_1}(x, x')$ at x. If g_1 and g_2 have the same lens data on $\widetilde M$, we have $S_{\widetilde g_2}(x, v) \in S_{x'}\widetilde M$ and therefore

$$d_{\widetilde g_2}(x, x') = \ell_{\widetilde g_2}(x, v) = \ell_{\widetilde g_1}(x, v) = d_{\widetilde g_1}(x, x').$$

Conversely, assume that $d_{\widetilde g_1} = d_{\widetilde g_2}$ on $\partial\widetilde M$. All we need to show is that $\widetilde g_1$ and $\widetilde g_2$ have the same scattering maps. Consider a point $(x, v) \in \partial SM$ that does not belong to the incoming tail of $\widetilde g_1$, and set $(x', v_1) := S_{\widetilde g_1}(x, v)$. Let v_2 be the unit tangent vector to $\alpha_{\widetilde g_2}(x, x')$ at x'. For $i = 1, 2$, we define the smooth functions $\zeta_i := d_{\widetilde g_i}(x, \cdot)$ and $\beta_i := \zeta_i|_{\partial\widetilde M}$. We denote by $\nabla^{\widetilde g_i}$ the gradient operator with respect to the Riemannian metric $\widetilde g_i$, and, by abuse of notation, also the gradient operator with respect to the Riemannian metric induced by $\widetilde g_i$ on $\partial\widetilde M$. By our assumptions, $\beta_1 = \beta_2$ and therefore $\nabla^{\widetilde g_1}\beta_1 = \nabla^{\widetilde g_2}\beta_2$. By Gauss Lemma, $\nabla^{\widetilde g_i}\zeta_i(x') = v_i$. Since $\nabla^{\widetilde g_i}\beta_i$ is the $\widetilde g_i$-orthogonal projection of $\nabla^{\widetilde g_i}\zeta_i$ onto $T(\partial\widetilde M)$, and since $v_1, v_2 \in T_{x'}\widetilde M$ are outward-pointing unit tangent vectors, we conclude that $v_1 = \nabla^{\widetilde g_1}\zeta_1(x') = \nabla^{\widetilde g_2}\zeta_2(x') = v_2$. Switching the roles of x and x', the same argument shows that the tangent vector to $\alpha_{\widetilde g_2}(x, x')$ at x is v, and therefore $S_{\widetilde g_1}(x, v) = S_{\widetilde g_2}(x, v)$. □

By applying an argument due to Croke [Cro91, Theorem C] we get the following result.

Lemma 2.5. *Let M be a compact surface with smooth boundary, and g_1, g_2 two Riemannian metrics on M without conjugate points, such that the boundary ∂M is strictly convex for each one of them, and $g_2 = e^{2\omega}g_1$ for some $\omega \in C^\infty(M)$ vanishing at ∂M. Assume that the trapped sets of the geodesic flows of g_1 and g_2 have zero measure for μ_{g_1} and μ_{g_2} respectively. If g_1 and g_2 have the same marked boundary distance function, then $g_1 = g_2$.*

Proof. Notice that g_1 and g_2 coincides on ∂M, and they both define the same inward-pointing unit vector field ν and the same measure $\mu_{g_1,\nu} = \mu_{g_2,\nu}$ on ∂M. We denote by φ_t^i the geodesic flow of the metric g_i. Since $d_{\widetilde{g}_1} = d_{\widetilde{g}_2}$, Lemma 2.4 implies that $(S_{\widetilde{g}_1}, \ell_{\widetilde{g}_1}|_{\partial_- S\widetilde{M}}) = (S_{\widetilde{g}_2}, \ell_{\widetilde{g}_2}|_{\partial_- S\widetilde{M}})$. In particular, $S_{g_1} = S_{g_2}$ and $\ell_{g_1}|_{\partial_- SM} = \ell_{g_2}|_{\partial_- SM}$. We denote by $\Gamma_\pm^i$ the incoming/outgoing tails of the geodesic flow of g_i. Notice that $\Gamma_\pm^1 \cap \partial_- SM = \Gamma_\pm^2 \cap \partial_- SM$. By (2.2), these tails have measure zero with respect to the corresponding Liouville measures. Therefore, Santaló's formula implies

$$\begin{aligned} \mathrm{Vol}_{g_1}(M) &= \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{\partial_- SM \setminus \Gamma_-^1} \ell_{g_1}(x,v) d\mu_{g_1,\nu}(x,v) \\ &= \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{\partial_- SM \setminus \Gamma_-^2} \ell_{g_2}(x,v) d\mu_{g_2,\nu}(x,v) \\ &= \mathrm{Vol}_{g_2}(M) \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} \mathrm{Vol}(S^{n-1}) \int_M e^{\omega(x)} d\mathrm{vol}_{g_1}(x) &= \int_{SM} e^{\omega(x)} d\mu_{g_1}(x,v) \\ &= \int_{SM} |v|_{g_2} d\mu_{g_1}(x,v) \\ &= \int_{\partial_- SM \setminus \Gamma_-^1} \int_0^{\ell_{g_1}(x,v)} |\varphi_t^1(x,v)|_{g_2} dt \, d\mu_{g_1,\nu}(x,v). \end{aligned} \tag{2.22}$$

Consider $(x,v) \in \partial_- SM \setminus \Gamma_-^1$ and, for $i = 1,2$, consider the geodesic

$$\gamma_i : [0, \ell_{g_i}(x,v)] \to M, \qquad (\gamma_i(t), \dot\gamma_i(t)) = \varphi_t^i(x,v).$$

Since $S_{\widetilde{g}_1} = S_{\widetilde{g}_2}$, we have $\gamma_1(\ell_{g_1}(x,v)) = \gamma_2(\ell_{g_2}(x,v))$, and the geodesics γ_1 and γ_2 are homotopic via a homotopy that fixes the endpoints. Therefore, by Lemma 2.2, we infer

$$\int_0^{\ell_{g_1}(x,v)} |\varphi_t^1(x,v)|_{g_2} dt = \mathrm{length}_{g_2}(\gamma_1) \geq \mathrm{length}_{g_2}(\gamma_2) = \ell_{g_2}(x,v) = \ell_{g_1}(x,v).$$

This, together with (2.21) and (2.22), implies

$$\begin{aligned} \mathrm{Vol}(S^{n-1}) \int_M e^{\omega(x)} d\mathrm{vol}_{g_1}(x) &\geq \int_{\partial_- SM \setminus \Gamma_-^1} \ell_{g_1}(x,v) \, d\mu_{g_1,\nu}(x,v) \\ &= \mathrm{Vol}(S^{n-1}) \, \mathrm{Vol}_{g_1}(M) \\ &= \mathrm{Vol}(S^{n-1}) \, \mathrm{Vol}_{g_2}(M). \end{aligned} \tag{2.23}$$

But then the Hölder inequality

$$\mathrm{Vol}_{g_2}(M)^{1/2} \, \mathrm{Vol}_{g_1}(M)^{1/2} \geq \int_M e^{\omega(x)} d\mathrm{vol}_{g_1}(x)$$

is satisfied as an equality, which implies $\omega \equiv 0$. ∎

Proof of Theorem 1. Let g_1 be a Riemannian metric as in the statement. By Proposition 2.1, there exists $\delta > 0$ such that, if a Riemannian metric g_2 on M satisfies $\|g_2 - g_1\|_{C^2} < \delta$, then g_2 satisfies the same properties as g_1 in the statement of Theorem 1. Assume further that g_1 and g_2 have the same marked boundary distance. In particular, these metrics have the same boundary distance, and therefore Lemmas 2.3 and 2.4 imply that they coincide on all points of ∂M and that their scattering maps are the same. By [Gui14, Theorem 3], there exists a diffeomorphism $\phi : M \to M$ which is the identity on ∂M and such that $\phi^* g_2 = e^{2\eta} g_1$ for some smooth function $\eta \in C^\infty(M)$ satisfying $\eta|_{\partial M} = 0$. We want to show that, up to reducing $\delta > 0$, we have $\eta \equiv 0$.

We claim that, for each $\epsilon > 0$, there is $\delta > 0$ small enough such that the diffeomorphism ϕ as above satisfies $\|\phi^{-1} - \mathrm{Id}\|_{C^2(M)} < \epsilon$. Indeed, if we set $q := g_1 - g_2$, then $(\phi^{-1})^* g_1 = e^{-2\eta\circ\phi^{-1}}(g_1 - q)$. Therefore, the map $\phi^{-1} : M \to M$ is quasi-conformal, equal to the identity on the boundary ∂M, and with Beltrami coefficient $\mu_0 \in C^\infty(M; \kappa^{-1} \otimes \overline{\kappa})$ having norm $\|\mu_0\|_{C^2(M)} \leq C\|q\|_{C^2(M)}$ for some $C > 0$ depending only on (M, g_1). Here, κ denotes the canonical bundle of (M, g_1). By [ES70, Section 3], the conformal structure of the interior of (M, g_1) can be realized as $\Gamma\backslash\mathbb{H}^2$ for some convex co-compact Fuchsian group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$, where $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ is the hyperbolic space. We denote by $\Omega \subset \mathbb{R} \cup \{\infty\}$ the set of discontinuity of Γ. We can assume that Ω contains $\{0, 1, \infty\}$. The discrete group Γ acts properly discontinuously on $\mathbb{H}^2 \cup \Omega$ by holomorphic transformations. The universal cover $\widetilde{M}$ can be identified with $\mathbb{H}^2 \cup \Omega$, and the fundamental group $\pi_1(M)$ can be identified with Γ. There is a holomorphic covering map $\pi : \mathbb{H}^2 \to M^\circ$, and the pull-back π^* is a homeomorphism of the smooth conformal structures on M to the set of Γ-equivariant smooth Beltrami differentials on $\mathbb{H}^2 \cup \Omega$. The Beltrami differential μ_0 lifts to a smooth Beltrami differential on $\mathbb{H}^2 \cup \Omega$ which is Γ-equivariant. The continuity theorem in [ES70, Section 8] implies that, for each Beltrami differential μ on $\mathbb{H}^2 \cup \Omega$ with $\|\mu\|_{L^\infty} < 1$, there is a unique smooth quasiconformal map $\Phi_\mu : \mathbb{H}^2 \cup \Omega \to \mathbb{H}^2 \cup \Omega$ that fixes $\{0, 1, \infty\}$, satisfies the Beltrami equation $\partial_{\bar{z}}\Phi_\mu = \mu\partial_z\Phi_\mu$, and such that the map $\mu \mapsto \Phi_\mu$ is continuous; here, we have equipped the domain of this map with the uniform C^2-topology on compact sets, and its codomain with the uniform C^2-topology on compact sets. We recall that Φ_μ is a solution of the Beltrami equation if and only if it is a conformal transformation of $\mathbb{H}^2 \cup \Omega$ equipped with the conformal structure $|dz + \mu d\bar{z}|^2$ to $\mathbb{H}^2 \cup \Omega$ equipped with the conformal structure induced by $\mathbb{C}$. By applying this result to the lift $\widetilde{\mu}_0$ of the Beltrami differential μ_0 to $\mathbb{H}^2 \cup \Omega$, we see by uniqueness that $\Phi_{\widetilde{\mu}_0}$ is equal to the lift of ϕ^{-1} to $\mathbb{H}^2 \cup \Omega$. The group Γ has a compact fundamental domain in $\mathbb{H}^2 \cup \Omega$, and therefore uniform C^2 estimates on M follow from C^2 estimates on compact sets of $\mathbb{H}^2 \cup \Omega$. We deduce that $\|\phi^{-1} - \mathrm{Id}\|_{C^2(M)} \leq C\|\mu_0\|_{C^2(M)}$, which implies our claim.

We recall that any map $M \to M$ that is sufficiently C^0-close to the identity and fixes the boundary ∂M is actually homotopic to the identity through maps that fix the boundary ∂M. Therefore, up to choosing $\epsilon > 0$ small enough above, the diffeomorphism ϕ is homotopic to the identity through maps that fix the endpoints. If α is any geodesic of $(M, e^{2\eta}g_1)$ that joins two endpoints in ∂M, the curve $\phi \circ \alpha$ is the corresponding geodesic of (M, g_2) joining the same endpoints. Since α and $\phi \circ \alpha$ are homotopic with fixed endpoints, and since g_1 and g_2 have the same marked boundary distance, g_1 and $e^{2\eta}g_1 = \phi^* g_2$ have the same marked boundary distance as well. By Lemma 2.5, we conclude that $\eta = 0$. $\square$

3. The negative curvature case using the method of Otal

3.1. The Liouville measure. Let (M, g) be a compact surface with strictly convex boundary and no conjugate points and, for now, we simply assume that the trapped set K has zero Liouville measure, i.e. $\mu_g(K) = 0$. We denote by $\widetilde{g} = \pi^* g$ the lift of the Riemannian metric to the universal covering $\pi : \widetilde{M} \to M$. By (2.2) we have that $\mu_g(\Gamma_- \cup \Gamma_+) = 0$, and thus the tails $\widetilde{\Gamma}_\pm \subset S\widetilde{M}$ of the geodesic flow of $(\widetilde{M}, \widetilde{g})$ have zero Liouville measure as well. Lemma 2.2 implies that there exists a unique geodesic joining each pair of points $x, x' \in M$ in each homotopy class of curves with endpoints x and x'. Equivalently, for each pair of points $x, x' \in \widetilde{M}$, there is a unique geodesic $\alpha_{\widetilde{g}}(x, x') : [0, d_{\widetilde{g}}(x, x')] \to \widetilde{M}$ going from x to x'. We consider the space

$$\mathcal{G} := (\partial\widetilde{M} \times \partial\widetilde{M}) \setminus \text{diag},$$

which we will see as the space of geodesics on $(\widetilde{M}, \widetilde{g})$ by means of the map $(x, x') \mapsto \alpha_{\widetilde{g}}(x, x')$. We denote by $\mathcal{M}$ the space of Borel measures on $\mathcal{G}$ invariant by the involution $(x, x') \mapsto (x', x)$.

Consider the open set

$$U := \big\{(x, x', t) \in \mathcal{G} \times (0, \infty) \;\big|\; 0 < t < d_{\widetilde{g}}(x, x')\big\},$$

and the diffeomorphism

$$\psi : U \to S\widetilde{M} \setminus (\widetilde{\Gamma}_+ \cup \widetilde{\Gamma}_-), \qquad \psi(x, x', t) = (\alpha_{\widetilde{g}}(x, x')(t), \partial_t \alpha_{\widetilde{g}}(x, x')(t)).$$

We denote by $\widetilde{\lambda}$ the contact form on $S\widetilde{M}$. We recall that the Liouville measure $\mu_{\widetilde{g}} = |\widetilde{\lambda} \wedge d\widetilde{\lambda}|$ is invariant by the geodesic flow $\widetilde{\varphi}_t$ of $(\widetilde{M}, \widetilde{g})$. If $\widetilde{X}$ denotes the geodesic vector field on $S\widetilde{M}$, we have $\psi_* \partial_t = \widetilde{X}$, and therefore

$$i_{\partial_t}(\psi^*(\widetilde{\lambda} \wedge d\widetilde{\lambda})) = \psi^*(i_{\widetilde{X}}(\widetilde{\lambda} \wedge d\widetilde{\lambda})) = \psi^*(d\widetilde{\lambda}). \tag{3.1}$$

Moreover, since $i_{\widetilde{X}} d\widetilde{\lambda} = 0$, we have

$$\mathcal{L}_{\partial_t} \psi^*(d\widetilde{\lambda}) = d(i_{\partial_t} \psi^*(d\widetilde{\lambda})) = d(\psi^* i_{\widetilde{X}} d\widetilde{\lambda}) = 0. \tag{3.2}$$

Equations (3.1) and (3.2) allow to define a measure $\eta_{\widetilde{g}} \in \mathcal{M}$ satisfying

$$\psi^* d\mu_{\widetilde{g}} = d\eta_{\widetilde{g}} \otimes dt. \tag{3.3}$$

With a slight abuse of terminology, we will call $\eta_{\widetilde{g}}$ the *Liouville measure* on $\mathcal{G}$ associated with the Riemannian metric $\widetilde{g}$. Let us provide a useful expression of the Liouville measure in a particular coordinate system. For each pair of distinct points $x, x' \in \widetilde{M}$, we denote by $\mathcal{F}(x, x') \subset \mathcal{G}$ the open subset of those $(y, y') \in \mathcal{G}$ whose associated geodesic $\alpha_{\widetilde{g}}(y, y')$ has a positive transverse intersection with $\alpha_{\widetilde{g}}(x, x')$. Notice that $\mathcal{F}(x, x')$ does not depend on the Riemannian metric $\widetilde g$ whenever x and x' belong to the boundary $\partial\widetilde M$. Consider the open set

$$V := \big\{(\tau,\theta)\in(0,d_{\widetilde g}(x,x'))\times(0,\pi)\ \big|\ \ell_{\widetilde g}(\alpha(\tau),R_\theta\dot\alpha(\tau))+\ell_{\widetilde g}(\alpha(\tau),-R_\theta\dot\alpha(\tau))<\infty\big\},$$

where $\alpha(\tau):=\alpha_{\widetilde g}(x,x')(\tau)$, and R_θ denotes the $+\theta$ rotation in the fibers of $S\widetilde M$. We define a diffeomorphism

$$\phi: V\to\mathcal F(x,x'),\qquad \phi(\tau,\theta):=(y,y'),$$

where y,y' are the endpoints of the geodesic passing trough $\alpha(\tau)$ and tangent to $R_\theta\dot\alpha(\tau)$:

$$\begin{aligned} y' &= \pi_0\circ\widetilde\varphi_{\ell_{\widetilde g}(\alpha(\tau),R_\theta\dot\alpha(\tau))}(\alpha(\tau),R_\theta\dot\alpha(\tau)),\\ y &= \pi_0\circ\widetilde\varphi_{\ell_{\widetilde g}(\alpha(\tau),-R_\theta\dot\alpha(\tau))}(\alpha(\tau),-R_\theta\dot\alpha(\tau)).\end{aligned}$$

Here, $\pi_0: S\widetilde M\to\widetilde M$ denotes the base projection.

Lemma 3.1. *The open set V has full measure in $(0,d_{\widetilde g}(x,x'))\times(0,\pi)$, and*

$$\phi^*\eta_{\widetilde g}=\sin(\theta)d\tau\,d\theta. \tag{3.4}$$

Proof. Let W be the open set of points $(\tau,\theta,t)\in V\times\mathbb R$ such that $\widetilde\varphi_t(\alpha(\tau),R_\theta\dot\alpha(\tau))$ belongs to the interior of $S\widetilde M$. We define the maps

$$\begin{aligned} \psi_1: W\to S\widetilde M,&\qquad \psi_1(\tau,\theta,t):=\widetilde\varphi_t(\alpha(\tau),R_\theta\dot\alpha(\tau)),\\ \psi_2: W\to U,&\qquad \psi_2(\tau,\theta,t):=(\phi(\tau,\theta),t+\ell_{\widetilde g}(\alpha(\tau),-R_\theta\dot\alpha(\tau))),\end{aligned}$$

which are diffeomorphisms onto their images. Notice that $\psi_1=\psi\circ\psi_2$.

The Liouville measure $\mu_{\widetilde g}$ induces a measure on the restriction $S\widetilde M|_\alpha$ of the unit tangent bundle along the geodesic $\alpha=\alpha_{\widetilde g}(x,x')$, given by

$$d\mu_{\widetilde g,\alpha}(y,v):=|\widetilde g_y(v,R_{\pi/2}\dot\alpha(\tau_y))|\,d\mathrm{vol}_\alpha(y)\,dS_y(v),$$

where $\tau_y\in(0,d_{\widetilde g}(x,x'))$ is such that $\alpha(\tau_y)=y$, $d\mathrm{vol}_\alpha$ is the Riemannian volume induced by $\widetilde g$ on α, and dS_y is the volume measure on the fiber $S_y\widetilde M$ induced by $\widetilde g$. We set $Z:=\psi_1(W)$. By Santaló's formula, for each $f\in C^\infty_c(Z)$ we have

$$\begin{aligned}\int_Z f\,d\mu_{\widetilde g} &= \int_{S\widetilde M|_\alpha}\int_{-\ell_{\widetilde g}(y,-v)}^{\ell_{\widetilde g}(y,v)} f(\widetilde\varphi_t(y,v))\,dt\,d\mu_{\widetilde g,\alpha}(y,v)\\ &= \int_0^{d_{\widetilde g}(x,x')}\int_0^\pi\int_{-\ell_{\widetilde g}(\alpha(\tau),-R_\theta\dot\alpha(\tau))}^{\ell_{\widetilde g}(\alpha(\tau),R_\theta\dot\alpha(\tau))} f(\widetilde\varphi_t(\alpha(\tau),R_\theta\dot\alpha(\tau)))\sin(\theta)\,dt\,d\theta\,d\tau,\end{aligned}$$

and therefore $\psi_1^*d\mu_{\widetilde g}=\sin(\theta)d\tau\,d\theta\,dt$. This, together with equation (3.3), implies that $\psi_2^*(\eta_{\widetilde g}\otimes dt)=\sin(\theta)d\tau\,d\theta\,dt$, which in turn implies (3.4). Since the incoming and outgoing tails $\widetilde\Gamma_\pm$ have zero Liouville measure, another application of Santaló's formula along the same line as above implies that V has full measure in $(0,d_{\widetilde g}(x,x'))\times(0,\pi)$. □

An immediate consequence of Lemma 3.1 is that

$$\eta_{\widetilde g}(\mathcal F(x,x'))=\int_0^\pi\int_0^{d_{\widetilde g}(x,x')}\sin(\theta)d\tau\,d\theta=2d_{\widetilde g}(x,x'),\qquad \forall x,x'\in\widetilde M. \tag{3.5}$$

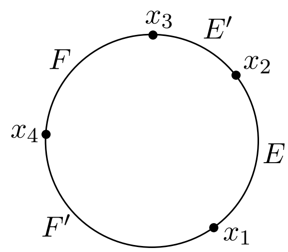

FIGURE 1. The circle containing $\partial\widetilde{M}$. The sets E, E', F, F' are not actually intervals in the circle, but are the intersection of the intervals drawn with $\partial\widetilde{M}$.

In particular, the distance function $d_{\widetilde{g}}$ is completely determined by (and actually equivalent to) the knowledge of the Liouville measure of every set $\mathcal{F}(x, x')$. In Otal's terminology [Ota90a, Ota90b], the quantity $\eta_{\widetilde{g}}(\mathcal{F}(x, x'))$ is the *intersection number* of the Liouville measure with the Dirac measure supported on the geodesic $\alpha_{\widetilde{g}}(x, x')$. These intersection numbers allow to recover the Liouville measure, as it follows from the next statement.

Lemma 3.2. *Two measures $\mu, \mu' \in \mathcal{M}$ are such that $\mu(\mathcal{F}(x, x')) = \mu'(\mathcal{F}(x, x'))$ for all distinct points $x, x' \in \partial\widetilde{M}$ if and only if $\mu = \mu'$.*

Proof. We recall that the boundary $\partial\widetilde{M}$ is homeomorphic to a countable union of real lines embedded in the circle S^1. We consider two subsets $E, F \subset \partial\widetilde{M}$, each one being the intersection of $\partial\widetilde{M}$ with a compact interval in S^1. We set

$$\mathcal{G}_{E,F} := (E \times F) \setminus \text{diag}.$$

All we have to prove is that $\mu(\mathcal{G}_{E,F}) = \mu'(\mathcal{G}_{E,F})$. The subsets E and F may overlap. By the additivity property of measures, it is enough to consider two cases: $E = F$ or $E \cap F = \varnothing$.

Let us first consider the case where E and F are disjoint, or intersects at most at one boundary point. We denote by x_1, x_2 the points in the boundary of E ordered according to the orientation of E, and analogously by x_3, x_4 the points in the boundary of F. We denote by $E', F' \subset \partial\widetilde{M}$ the complementary regions (which may be empty), as in Figure 1. We have

$$\begin{aligned}
\mu(\mathcal{F}(x_1, x_3)) &= \mu(\mathcal{G}_{E,F}) + \mu(\mathcal{G}_{E,F'}) + \mu(\mathcal{G}_{E',F}) + \mu(\mathcal{G}_{E',F'}),\\
\mu(\mathcal{F}(x_4, x_2)) &= \mu(\mathcal{G}_{E,F}) + \mu(\mathcal{G}_{E,E'}) + \mu(\mathcal{G}_{F',F}) + \mu(\mathcal{G}_{F',E'}),\\
\mu(\mathcal{F}(x_2, x_3)) &= \mu(\mathcal{G}_{E',E}) + \mu(\mathcal{G}_{E',F'}) + \mu(\mathcal{G}_{E',F}),\\
\mu(\mathcal{F}(x_1, x_4)) &= \mu(\mathcal{G}_{E,F'}) + \mu(\mathcal{G}_{E',F'}) + \mu(\mathcal{G}_{F,F'}).
\end{aligned}$$

Since μ is invariant under the involution $(x,x')\mapsto(x',x)$, we have $\mu(\mathcal{G}_{E',F})=\mu(\mathcal{G}_{F,E'})$, $\mu(\mathcal{G}_{E',F'})=\mu(\mathcal{G}_{F',E'})$, and $\mu(\mathcal{G}_{F',F})=\mu(\mathcal{G}_{F,F'})$. Therefore

$$\begin{aligned}\mu(\mathcal{G}_{E,F}) &= \tfrac{1}{2}\big(\mu(\mathcal{F}(x_1,x_3))+\mu(\mathcal{F}(x_4,x_2))-\mu(\mathcal{F}(x_2,x_3))-\mu(\mathcal{F}(x_1,x_4))\big)\\ &= \tfrac{1}{2}\big(\mu'(\mathcal{F}(x_1,x_3))+\mu'(\mathcal{F}(x_4,x_2))-\mu'(\mathcal{F}(x_2,x_3))-\mu'(\mathcal{F}(x_1,x_4))\big)\\ &= \mu'(\mathcal{G}_{E,F}).\end{aligned}$$

Now, consider the case where $E=F$. We parametrize the circle S^1 containing $\partial\widetilde{M}$ from $\theta=0$ to $\theta=2\pi$ in such a way that $E=[0,\theta_0]\cap\partial\widetilde{M}$ for some $\theta_0\in[0,2\pi]$. We split E in two parts of equal size, so that $E=E_1\cup E_2$ for $E_1=[0,\theta_0/2]\cap\partial\widetilde{M}$ and $E_2=[\theta_0/2,\theta_0]\cap\partial\widetilde{M}$. We apply the same splitting iteratively, so that $E_1=E_{11}\cup E_{12}$, $E_2=E_{21}\cup E_{22}$, and more generally $E_{i_0\ldots i_n}=E_{i_0\ldots i_{n-1}1}\cup E_{i_0\ldots i_{n-1}2}$. Notice that

$$\mathcal{G}_{E,E}=\bigcup_{n=1}^{\infty}\ \bigcup_{i_1,\ldots,i_{n-1}\in\{1,2\}}\big(\mathcal{G}_{E_{i_0\ldots i_{n-1}1},E_{i_0\ldots i_{n-1}2}}\cup\mathcal{G}_{E_{i_0\ldots i_{n-1}2},E_{i_0\ldots i_{n-1}1}}\big).$$

By the σ-additivity of measures, we have

$$\begin{aligned}\mu(\mathcal{G}_{E,E}) &= \sum_{n=1}^{\infty}\ \sum_{i_1,\ldots,i_{n-1}\in\{1,2\}}\big(\mu(\mathcal{G}_{E_{i_0\ldots i_{n-1}1},E_{i_0\ldots i_{n-1}2}})+\mu(\mathcal{G}_{E_{i_0\ldots i_{n-1}2},E_{i_0\ldots i_{n-1}1}})\big)\\ &= \sum_{n=1}^{\infty}\ \sum_{i_1,\ldots,i_{n-1}\in\{1,2\}}\big(\mu'(\mathcal{G}_{E_{i_0\ldots i_{n-1}1},E_{i_0\ldots i_{n-1}2}})+\mu'(\mathcal{G}_{E_{i_0\ldots i_{n-1}2},E_{i_0\ldots i_{n-1}1}})\big)\\ &= \mu'(\mathcal{G}_{E,E}).\end{aligned}$$ □

The following is an immediate consequence of equation (3.5) and Lemma 3.2.

Corollary 3.3. *Let M be a compact surface with smooth boundary, and g_1,g_2 two Riemannian metrics with no conjugate points, strictly convex boundary ∂M, and trapped set of zero Liouville measure. Then g_1 and g_2 have the same marked boundary distance if and only if they have the same Liouville measure $\eta_{\widetilde{g}_1}=\eta_{\widetilde{g}_2}$.* □

3.2. **The rigidity result.** Let M be a compact surface with smooth boundary. On M, we consider a negatively-curved Riemannian metric g_1 that makes ∂M strictly convex; we then consider another Riemannian metric g_2 with no conjugate points, that makes ∂M strictly convex, and such that the trapped set of its geodesic flow has zero Liouville measure. Since (M,g_1) has negative curvature, it does not have conjugate points and the trapped set of its geodesic flow has zero Liouville measure (see [Gui14, Section 2.4]). Therefore, g_1 and g_2 satisfy the assumptions of Corollary 3.3. Assume that they also have the same marked boundary distance $d_{\widetilde{g}_1}=d_{\widetilde{g}_2}$, and thus the same Liouville measures $\eta_{\widetilde{g}_1}=\eta_{\widetilde{g}_2}$ according to Corollary 3.3. We denote by SM and $S\widetilde{M}$ the unit tangent bundle for g_1 over M and for $\widetilde{g}_1$ over $\widetilde{M}$, and by $\widetilde{\Gamma}_\pm$ the incoming/outgoing tails of the geodesic flow of $(\widetilde{M},\widetilde{g}_1)$. For each $(x,v)\in S\widetilde{M}$, let $R_\theta v\in S_x\widetilde{M}$ be the vector obtained after a counterclockwise rotation of v in the fiber by an angle $\theta\in[0,\pi]$, where the angle is measured by $\widetilde{g}_1$. Consider the two geodesics of $\widetilde{g}_1$ of $S\widetilde{M}$ intersecting at x and with tangent vectors v and $R_\theta v$. If both (x,v) and $(x,R_\theta v)$ are not in $\widetilde{\Gamma}_- \cup \widetilde{\Gamma}_+$, these geodesics can be written as $\alpha_{\widetilde{g}_1}(z,z')$ and $\alpha_{\widetilde{g}_1}(w,w')$ for some $z,z',w,w' \in \partial\widetilde{M}$. The corresponding geodesics $\alpha_{\widetilde{g}_2}(z,z')$ and $\alpha_{\widetilde{g}_2}(w,w')$ for the metric $\widetilde{g}_2$ intersect at some point x'' with an angle $\widetilde{\theta}''(x,v,\theta)$. This defines a function

$$\widetilde{\theta}'' : W_1 \to [0,\pi],$$

where W_1 is the open set

$$W_1 := \big\{(x,v,\theta) \in S\widetilde{M} \times [0,\pi] \ \big|\ (x,v),(x,R_\theta v) \notin (\widetilde{\Gamma}_- \cup \widetilde{\Gamma}_+)\big\}.$$

The complement $(S\widetilde{M} \times [0,\pi]) \setminus W_1$ has zero measure with respect to $\mu_{\widetilde{g}_1} \otimes d\theta$. Let $Z \subset \mathcal{G} \times \mathcal{G}$ be the open set of quadruples (z,z',w,w') whose associated geodesics $\alpha_{\widetilde{g}_1}(z,z')$ and $\alpha_{\widetilde{g}_1}(w,w')$ are distinct and intersect at one point. We define the diffeomorphism

$$\kappa_1 : W_1 \to Z, \qquad \kappa_1(x,v,\theta) = (z,z',w,w'),$$

where, as before, $\alpha_{\widetilde{g}_1}(z,z')$ and $\alpha_{\widetilde{g}_1}(w,w')$ are the geodesic passing through the point x with tangent vectors v and $R_\theta v$ respectively. For the Riemannian metric $\widetilde{g}_2$ we can define an analogous diffeomorphism $\kappa_2 : W_2 \to Z$. Notice that the function $\widetilde{\theta}''$ appears in the composition $\kappa_2^{-1} \circ \kappa_1$, which indeed has the form

$$\kappa_2^{-1} \circ \kappa_1(x,v,\theta) = (x''(x,v,\theta), v''(x,v,\theta), \widetilde{\theta}''(x,v,\theta)). \tag{3.6}$$

Therefore $\widetilde{\theta}''$ is a smooth function.

Lemma 3.4. *For all $(x,v) \in S\widetilde{M}$ and $\theta_1,\theta_2 \in [0,\pi]$ such that $\theta_1+\theta_2 \in [0,\pi]$, $(x,v,\theta_1) \in W_1$, and $(x,R_{\theta_1}v,\theta_2) \in W_1$, we have*

$$\widetilde{\theta}''(x,v,\theta_1) + \widetilde{\theta}''(x,R_{\theta_1}v,\pi-\theta_1) = \pi, \tag{3.7}$$

$$\widetilde{\theta}''(x,v,\theta_1) + \widetilde{\theta}''(x,R_{\theta_1}v,\theta_2) \leq \widetilde{\theta}''(x,v,\theta_1+\theta_2). \tag{3.8}$$

Moreover, if we set

$$\begin{aligned}(z,z',w,w') &:= \kappa_1(x,v,\theta_1),\\ (w,w',y,y') &:= \kappa_1(x,R_{\theta_1}v,\theta_2),\end{aligned} \tag{3.9}$$

the inequality (3.8) is satisfied as an equality if and only if the three geodesics $\alpha_{\widetilde{g}_2}(z,z')$, $\alpha_{\widetilde{g}_2}(w,w')$, and $\alpha_{\widetilde{g}_2}(y,y')$ intersect at one point of $\widetilde{M}$.

Proof. Equation (3.7) follows from the very definition of $\widetilde{\theta}''$. Consider the three pairs $(z,z'),(w,w'),(y,y') \in \mathcal{G}$ defined by (3.9), and the corresponding geodesics $\alpha_{\widetilde{g}_2}(z,z')$, $\alpha_{\widetilde{g}_2}(w,w')$, and $\alpha_{\widetilde{g}_2}(y,y')$ for the Riemannian metric $\widetilde{g}_2$. Some portions of these three geodesics are the edges of a geodesic triangle in $(\widetilde{M},\widetilde{g}_2)$ whose vertices are the mutual intersections among $\alpha_{\widetilde{g}_2}(z,z')$, $\alpha_{\widetilde{g}_2}(w,w')$, and $\alpha_{\widetilde{g}_2}(y,y')$. This triangle may degenerate to a single point x_0 if the three geodesics intersect at x_0. The interior angles of this geodesic triangle are precisely $\widetilde{\theta}''(x,v,\theta_1)$, $\widetilde{\theta}''(x,R_{\theta_1}v,\theta_2)$, and $\widetilde{\theta}''(x,R_{\theta_1+\theta_2}v,\pi-\theta_1-\theta_2)$. Since $\widetilde{g}_2$ has negative curvature, Gauss-Bonnet formula for geodesic polygons implies

$$\widetilde{\theta}''(x,v,\theta_1)+\widetilde{\theta}''(x,R_{\theta_1}v,\theta_2)+\widetilde{\theta}''(x,R_{\theta_1+\theta_2}v,\pi-\theta_1-\theta_2)\leq\pi. \tag{3.10}$$

This inequality is not strict if and only if the geodesic triangle is reduced to a point $x_0\in\widetilde{M}$, that is, if and only if the three geodesics $\alpha_{\widetilde{g}_2}(z,z')$, $\alpha_{\widetilde{g}_2}(w,w')$, and $\alpha_{\widetilde{g}_2}(y,y')$ intersect at x_0. Finally, (3.10) and (3.7) imply (3.8). □

Proof of Theorem 2. Notice that $\widetilde{\theta}''$ descends to $SM\times[0,\pi]$ as a measurable bounded map $\theta'':SM\times[0,\pi]\to[0,\pi]$ which is smooth on an open set of full measure. We introduce the average angle function

$$\Theta:[0,\pi]\to[0,\pi],\qquad \Theta(\theta):=\frac{1}{\mathrm{Vol}_{g_1}(SM)}\int_{SM}\theta''(x,v,\theta)\,d\mu_{g_1}(x,v),$$

which is continuous by Lebesgue theorem. Since $\theta''(x,v,0)=0$ and $\theta''(x,v,\pi)=\pi$ almost everywhere, we have

$$\Theta(0)=0,\qquad \Theta(\pi)=\pi. \tag{3.11}$$

Notice that the rotation R_θ, seen as a diffeomorphism of the unit tangent bundle SM, preserves the Liouville measure μ_{g_1}. Clearly, Equations (3.7) and (3.8) still hold if we replace $\widetilde{\theta}''$ with θ'', and if we integrate them on SM against μ_{g_1} we obtain

$$\Theta(\pi-\theta)=\pi-\Theta(\theta),\qquad \forall\theta\in[0,\pi], \tag{3.12}$$

$$\Theta(\theta_1+\theta_2)\geq\Theta(\theta_1)+\Theta(\theta_2),\qquad \forall\theta_1,\theta_2\in[0,\pi]\text{ with }\theta_1+\theta_2\in[0,\pi]. \tag{3.13}$$

If $f:[0,\pi]\to\mathbb{R}$ is a continuous convex function, we have Jensen's inequality

$$f(\Theta(\theta))\leq\frac{1}{\mathrm{Vol}_{g_1}(SM)}\int_{SM}f(\theta''(x,v,\theta))\,d\mu_{g_1}(x,v). \tag{3.14}$$

Actually, if f is strictly convex, this inequality is satisfied as an equality if and only if $\theta''(x,v,\theta)=\theta$ for almost all $(x,v)\in SM$. Integrating in θ, by Fubini's Theorem we get

$$\int_0^\pi f(\Theta(\theta))\sin(\theta)d\theta\leq\frac{1}{\mathrm{Vol}_{g_1}(SM)}\int_{SM}\int_0^\pi f(\theta''(x,v,\theta))\sin(\theta)d\theta\,d\mu_{g_1}(x,v).$$

We set

$$F(x,v):=\int_0^\pi f(\theta''(x,v,\theta))\sin(\theta)d\theta.$$

By applying Santaló's formula, we obtain

$$\int_{SM}F(x,v)\,d\mu_{g_1}=\int_{\partial_-SM}\int_0^{\ell_{g_1}(x,v)}F(\varphi_t(x,v))dt\,d\mu_{g_1,\nu}(x,v).$$

We fix two arbitrary distinct points $x, x' \in \partial\widetilde{M}$. For $i = 1, 2$, Lemma 3.1 gives a subset of full measure $V_i \subset (0, d_{\widetilde{g}_1}(x, x')) \times (0, \pi)$ and a diffeomorphisms $\phi_i : V_i \to \mathcal{F}(x, x')$ such that $\phi_i^* \eta_{\widetilde{g}_i} = \sin(\theta) d\tau\, d\theta$. We consider the composition $\phi_2^{-1} \circ \phi_1 : V_1 \to V_2$, which has the form

$$(\tau, \theta) \mapsto (\tau''(\tau, \theta), \widetilde{\theta}''(\alpha(\tau), \dot{\alpha}(\tau), \theta)),$$

where $\alpha(\tau) := \alpha_{\widetilde{g}_1}(x, x')$. Since $\eta_{\widetilde{g}_1} = \eta_{\widetilde{g}_2}$, Lemma 3.1 implies

$$(\phi_2^{-1} \circ \phi_1)^* \sin(\theta) d\tau\, d\theta = \phi_1^* \eta_{\widetilde{g}_2} = \phi_1^* \eta_{\widetilde{g}_1} = \sin(\theta) d\tau\, d\theta.$$

From this we infer that, for all $(x, v) \in \partial_- S\widetilde{M} \setminus \widetilde{\Gamma}_-$,

$$\begin{aligned}
\int_0^{\ell_{g_1}(\pi(x,v))} F(\varphi_\tau(\pi(x, v))) d\tau &= \int_0^{d_{\widetilde{g}_1}(x,x')} F(\pi(\widetilde{\varphi}_\tau(x, v))) d\tau \\
&= \int_0^{d_{\widetilde{g}_1}(x,x')} \int_0^\pi f(\widetilde{\theta}''(\alpha(\tau), \dot{\alpha}(\tau), \theta)) \sin(\theta) d\theta\, d\tau \\
&= \int_0^{d_{\widetilde{g}_1}(x,x')} \int_0^\pi f(\theta) \sin(\theta) d\theta\, d\tau \\
&= \ell_{g_1}(\pi(x, v)) \int_0^\pi f(\theta) \sin(\theta) d\theta,
\end{aligned}$$

where $v \in S_x\widetilde{M}$ denotes the tangent vector to the geodesic $\alpha_{\widetilde{g}_1}(x, x')$ at the starting point x. We integrate the previous equality over $\partial_- SM$ against $d\mu_{g_1,\nu}$. By applying Santaló's formula to the left-hand side, the volume computation (2.13) to the right-hand side, and finally Jensen's inequality (3.14) to the left-hand side, we obtain

$$\int_0^\pi f(\Theta(\theta)) \sin(\theta) d\theta \le \int_0^\pi f(\theta) \sin(\theta) d\theta.$$

Since this inequality must be satisfied for all continuous convex functions $f : [0, \pi] \to \mathbb{R}$, Equations (3.11), (3.12), and (3.13) readily imply that $\Theta(\theta) = \theta$ for all $\theta \in [0, \pi]$, see [Ota90a, Lemma 8]. Therefore, the inequality (3.8) is satisfied as an equality. By the "moreover" part of Lemma 3.4, whenever three geodesics $\alpha_{\widetilde{g}_1}(x, x')$, $\alpha_{\widetilde{g}_1}(y, y')$, $\alpha_{\widetilde{g}_1}(z, z')$ intersect at a single point x_0, the corresponding geodesics for the other Riemannian metric $\alpha_{\widetilde{g}_2}(x, x'), \alpha_{\widetilde{g}_2}(y, y'), \alpha_{\widetilde{g}_2}(z, z')$ must intersect in a single point, which we denote by $\widetilde{\psi}(x_0)$, as well. This defines a map $\widetilde{\psi} : \widetilde{M} \to \widetilde{M}$. Fix a point $x \in \widetilde{M}$. The domain W_1 of $\kappa_1 : W_1 \to Z$ is an open set with projection on the base given by the whole manifold $\widetilde{M}$. Consider the composition of diffeomorphisms $\kappa_2^{-1} \circ \kappa_1$, which has the form (3.6). For each $x \in U$, if $v \in T_x\widetilde{M}$ and $\theta \in [0, \pi]$ are such that both (x, v) and $(x, R_\theta v)$ are not in $\widetilde{\Gamma}_- \cup \widetilde{\Gamma}_+$, we have $\widetilde{\psi}(x) = x''(x, v, \theta)$. This shows that $\widetilde{\psi}$ is smooth.

For $x_0 \in \partial\widetilde{M}$, we can choose four points $y, y', z, z' \in \partial\widetilde{M}$ near x_0 so that, according to the orientation of $\partial\widetilde{M}$, the points are in the order y, z, x_0, y', z'. If both y, z' tend to x_0, then so do the points y', z and, by the strict convexity of $\partial\widetilde{M}$, the intersection

$x := \alpha_{\widetilde{g}_1}(y,y') \cap \alpha_{\widetilde{g}_1}(z,z')$. By the same reasoning $\widetilde{\psi}(x) = \alpha_{\widetilde{g}_2}(y,y') \cap \alpha_{\widetilde{g}_2}(z,z')$ converges to x_0 as well. This proves that $\widetilde{\psi}|_{\partial\widetilde{M}} = \mathrm{Id}$.

Clearly $\widetilde{\psi}$ maps every geodesic of $\widetilde{g}_1$ to the geodesic of $\widetilde{g}_2$ with the same endpoints. For $i = 1,2$ and for all distinct points $x, x' \in \widetilde{M}$, consider the open set $\mathcal{F}_i(x,x') \subset \mathcal{G}$ of those (y,y') whose associated geodesic $\alpha_{\widetilde{g}_i}(y,y')$ has a positive transverse intersection with $\alpha_{\widetilde{g}_i}(x,x')$. It is straighforward to see that

$$\mathcal{F}_1(x,x') = \mathcal{F}_2(\widetilde{\psi}(x), \widetilde{\psi}(x')).$$

Therefore, by Equation (3.5), we have

$$d_{\widetilde{g}_1}(x,x') = \tfrac{1}{2}\eta_{\widetilde{g}_1}(\mathcal{F}_1(\widetilde{\psi}(x),\widetilde{\psi}(x'))) = \tfrac{1}{2}\eta_{\widetilde{g}_2}(\mathcal{F}_2(\widetilde{\psi}(x),\widetilde{\psi}(x'))) = d_{\widetilde{g}_2}(\widetilde{\psi}(x),\widetilde{\psi}(x')).$$

This readily implies that $\widetilde{\psi}^*\widetilde{g}_2 = \widetilde{g}_1$. Indeed, for all $(x,v) \in SM$, if $\zeta : [0,\epsilon) \to \widetilde{M}$ is a smooth curve such that $\zeta(0) = x$ and $\dot\zeta(0) = v$, we have

$$\|v\|_{\widetilde{g}_1} = \lim_{t\to 0} \frac{d_{\widetilde{g}_1}(\zeta(0),\zeta(t))}{t} = \lim_{t\to 0} \frac{d_{\widetilde{g}_2}(\widetilde{\psi}(\zeta(0)),\widetilde{\psi}(\zeta(t)))}{t} = \|d\widetilde{\psi}(x)v\|_{\widetilde{g}_2}.$$

Finally, since the fundamental group $\pi_1(M)$ acts on $\widetilde{M}$ by isometries with respect to both $\widetilde{g}_1$ and $\widetilde{g}_2$, we have $\widetilde{\psi}(\gamma\cdot x) = \gamma\cdot\widetilde{\psi}(x)$ for each $\gamma \in \pi_1(M)$, and thus $\widetilde{\psi}$ descends to a smooth isometry $\psi : M \to M$ fixing the boundary. □

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