# CLOSED GEODESICS WITH LOCAL HOMOLOGY IN MAXIMAL DEGREE ON NON-COMPACT MANIFOLDS
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# CORRIGENDUM

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The proof of [AM18, Theorem 1.1, case $\overline{\mathrm{ind}}(\gamma) > 0$] contains a mistake, that nevertheless does not affect the validity of the theorem. We thank Egor Shelukhin and Vukasin Stojisavljevic for pointing out the mistake to the second author.

The correction consists of the following two points:

• In [AM18, Theorem 1.1(ii)], the assumption made automatically implies a stronger one, according to the following lemma.

**Lemma 0.1.** *Let $\gamma$ be a closed geodesic in a Riemannian manifold $(M, g)$. If*

$$\mathrm{ind}(\gamma^m) + \mathrm{nul}(\gamma^m) = m\,\overline{\mathrm{ind}}(\gamma) + \dim(M) - 1, \qquad \forall m \in \{1\} \cup \mathbb{P}, \tag{0.1}$$

*where $\mathbb{P}$ is an infinite subset of prime numbers, then $\mathrm{nul}(\gamma) = \mathrm{nul}(\gamma^m)$ for all $m \in \mathbb{N}$, and the identity (0.1) holds for all $m \in \mathbb{N}$.*

*Proof.* Bott's iteration theory implies that the Morse indices of the iterates of $\gamma$ satisfy

$$\mathrm{ind}(\gamma^m) + \mathrm{nul}(\gamma^m) \leq m\,\overline{\mathrm{ind}}(\gamma) + \dim(M) - 1, \qquad \forall m \in \mathbb{N}, \tag{0.2}$$

and if this inequality is not strict for a given $m$ then $\gamma^m$ has only 1 as Floquet multiplier, see e.g. [GH09, Section 6]. Since we are assuming that (0.2) is not strict for arbitrarily large prime numbers $m$, we infer that $\gamma$ has only 1 as Floquet multiplier. In particular, $\mathrm{nul}(\gamma^m) = \mathrm{nul}(\gamma)$ for all $m \in \mathbb{N}$, and Bott's function $\Lambda : S^1 \to \mathbb{N} \cup \{0\}$ associated to $\gamma$ satisfies $\Lambda(1) = \mathrm{ind}(\gamma)$ and $\Lambda(z) = \overline{\mathrm{ind}}(\gamma)$ for all $z \in S^1 \setminus \{1\}$. Therefore

$$\mathrm{ind}(\gamma^m) + \mathrm{nul}(\gamma^m) = \sum_{z \in \sqrt[m]{1}} \Lambda(z) + \mathrm{nul}(\gamma) = (m-1)\,\overline{\mathrm{ind}}(\gamma) + \mathrm{ind}(\gamma) + \mathrm{nul}(\gamma) \qquad \forall m \in \mathbb{N}.$$

This, together with our assumption $\mathrm{ind}(\gamma) + \mathrm{nul}(\gamma) = \overline{\mathrm{ind}}(\gamma) + \dim(M) - 1$, implies the lemma. ∎

• The arguments in Sections 5.1, 5.2, and 5.3 were stated for large enough $m \in \{1\} \cup \mathbb{P}$. In view of the previous point these arguments actually hold for large enough odd integers $m \in \mathbb{N}$. Indeed, Lemma 0.1 in the previous point implies that the equality (0.1) holds for all $m \in \mathbb{N}$, and $\mathrm{nul}(\gamma^m)$ is independent of $m \in \mathbb{N}$; moreover, by choosing $m$ to be odd, the difference $\mathrm{ind}(\gamma^m) - \mathrm{ind}(\gamma)$ is always even

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(this is a general fact that follows from Bott's iteration formula), and therefore the inclusion induces an injective homomorphism

$$C_*(E,\gamma) \xhookrightarrow{\text{incl}} C_*(E, S^1\cdot\gamma).$$

Once this is understood, the original proof of [AM18, Theorem 1.1, case $\overline{\mathrm{ind}}(\gamma)>0$], which takes about a page at the very end of [AM18, Section 5], should be replaced with the following one, whose central argument is due to Hingston [Hin97]:

*Proof of Theorem 1.1, case* $\overline{\mathrm{ind}}(\gamma)>0$. Let us assume without loss of generality that

$$\ell := \mathrm{length}(\gamma) = 1.$$

We prove the theorem by contradiction, by assuming that $(M,g)$ has only finitely many non-iterated closed geodesics

$$\gamma_1,...,\gamma_a,\gamma_{a+1},...,\gamma_{a+b},$$

with $\mathrm{length}(\gamma_i)=\ell_i\in\mathbb{R}\setminus\mathbb{Q}$ for all $i=1,...,a$ and $\mathrm{length}(\gamma_j)=\ell_j=p_j/q_j\in\mathbb{Q}$ for all $j=a+1,...,a+b$. Here, $p_j,q_j$ are natural numbers. If $a>0$, that is, there are closed geodesics of irrational length, we set $\ell_0:=\min\{\ell_1,...,\ell_a\}$, and

$$\delta_1 := \min\Big\{ |m_1\ell_i - m_2| \ \Big|\ m_1,m_2\in\mathbb{N} \text{ with } m_1\leq \tfrac{2(a+2)}{\ell_0},\ i=1,...,a\Big\} > 0.$$

We know that $b>0$, since at least the closed geodesic $\gamma$ has rational length, and we have

$$\begin{aligned}\delta_2 &:= \min\Big\{ \big|m_1\tfrac{p_i}{q_i} - m_2\big| \ \Big|\ m_1,m_2\in\mathbb{N} \text{ with } m_1\tfrac{p_i}{q_i}-m_2\neq 0,\ i=a+1,...,a+b\Big\}\\ &\geq \min\left\{\frac{1}{q_{a+1}},...,\frac{1}{q_{a+b}}\right\} > 0.\end{aligned}$$

Now, we fix $\epsilon\in(0,\min\{\delta_1,\delta_2,1\})$. By Lemma 5.5, for each odd integer $m>\max\{\overline{m}_\epsilon,\tilde m\}$ there exists a closed geodesic $\zeta_m\in\mathrm{crit}(E)$ such that

$$m < \mathrm{length}(\zeta_m) \leq m+\epsilon. \tag{0.3}$$

Each closed geodesic $\zeta_m$ must be of the form $\zeta_m=\gamma_i^\mu$ for some $i=i(m)\in\{1,...,a+b\}$ and $\mu=\mu(m)\in\mathbb{N}$. Notice that $i(m)\not\in\{a+1,...,a+b\}$, since the inequality (0.3) implies

$$0<\mu(m)\ell_{i(m)} - m < \epsilon < \delta_2.$$

Therefore $i(m)\in\{1,...,a\}$ for all odd integers $m\in\mathbb{N}$, that is, every geodesic $\gamma_{i(m)}$ has irrational length $\ell_{i(m)}\in\mathbb{R}\setminus\mathbb{Q}$. Since there are only $a$ such closed geodesics, we can find two integers $m_1,m_2$ both larger than $\max\{\overline{m}_\epsilon,\tilde m\}$ and such that $m_1<m_2\leq m_1+2(a+1)$ and $i:=i(m_1)=i(m_2)$. Equation (0.3) applied to these two integers gives

$$m_1<\mu(m_1)\ell_i\leq m_1+\epsilon, \qquad m_2<\mu(m_2)\ell_i\leq m_2+\epsilon. \tag{0.4}$$

Therefore

$$|(\mu(m_2)-\mu(m_1))\ell_i - (m_2-m_1)| \leq \epsilon < \delta_1,$$

which implies $\mu(m_2)-\mu(m_1) > \frac{2(a+2)}{\ell_0}$ according to the definition of $\delta_1$. However, this gives a contradiction, since the inequalities (0.4) imply

$$\mu(m_2)-\mu(m_1) < \frac{m_2-m_1+\epsilon}{\ell_i} < \frac{2(a+1)+1}{\ell_0}. \qquad \blacksquare$$

• Beside the above points, we realized that the assumption $\mathrm{nul}(\gamma^m) = \mathrm{nul}(\gamma)$ in Lemma 3.5 is superfluous, and can be removed. Corollary 3.6 can therefore be improved as follows.

**Corollary 3.6.** *Let $\gamma$ be a non-iterated isolated closed geodesic. For all positive integers $m$ such that $\mathrm{ind}(\gamma^m) - \mathrm{ind}(\gamma)$ is even, the inclusion induces an injective homomorphism*

$$C_*(E, \gamma^m) \xhookrightarrow{\text{incl}} C_*(E, S^1 \cdot \gamma^m).$$

*The difference $\mathrm{ind}(\gamma^m) - \mathrm{ind}(\gamma)$ is even if $m$ is odd.*

## References

[AM18] L. Asselle and M. Mazzucchelli, *Closed geodesics with local homology in maximal degree on non-compact manifolds*, Differential Geom. Appl. **58** (2018), 17–51.

[GH09] M. Goresky and N. Hingston, *Loop products and closed geodesics*, Duke Math. J. **150** (2009), no. 1, 117–209.

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