THE MULTIPLICITY PROBLEM FOR PERIODIC ORBITS OF MAGNETIC FLOWS ON THE 2-SPHERE

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To the memory of Abbas Bahri (1955–2016)

Abstract. We consider magnetic Tonelli Hamiltonian systems on the cotangent bundle of the 2-sphere, where the magnetic form is not necessarily exact. It is known that, on very low and on high energy levels, these systems may have only finitely many periodic orbits. Our main result asserts that almost all energy levels in a precisely characterized intermediate range (e_0, e_1) possess infinitely many periodic orbits. Such a range of energies is non-empty, for instance, in the physically relevant case where the Tonelli Lagrangian is a kinetic energy and the magnetic form is oscillating (in which case, $e_0 = 0$ is the minimal energy of the system).

1. Introduction

This paper is the last chapter of a work started in [AMP15] and further developed in [AMMP14, AB15a, AB15b, AM16] devoted to studying the multiplicity of periodic orbits on generic low energy levels in magnetic Tonelli Lagrangian systems on surfaces. Such a study was based on a generalization of Bangert's waist Theorem [Ban80, Theorem 4], classically formulated for geodesic flows on S^2, to the magnetic Tonelli setting. Roughly speaking, a waist is a non-constant periodic geodesic (resp. a periodic orbit in the Tonelli case) which minimizes the length (resp. the action) among nearby curves. The original waist Theorem says that a Riemannian 2-sphere possesses infinitely many closed geodesics provided it possesses a waist. Such a statement is a crucial ingredient for the proof that, indeed, every Riemannian 2-sphere possesses infinitely many closed geodesics [Ban93, Fra92, Hin93].

Let us introduce the general setting in which we will work. If M is a closed smooth manifold, a Tonelli Lagrangian $L : \mathrm{T}M \to \mathbb{R}$ is a smooth function whose restriction to any fiber of $\mathrm{T}M$ is superlinear with positive definite Hessian, see e.g. [Mat91, Fat08, Abb13]. A magnetic Tonelli system is a pair (L, σ), where $L : \mathrm{T}M \to \mathbb{R}$ is a Tonelli Lagrangian and σ is a closed 2-form on M, which we refer to as the magnetic form. If $\pi : \mathrm{T}^*M \to M$ denotes the projection of the cotangent bundle, the pair (L, σ) defines a flow on $\mathrm{T}M$ that is conjugated through the Legendre transformation $\partial_v L$ to the Hamiltonian flow on $(\mathrm{T}^*M, \mathrm{d}p \wedge \mathrm{d}q + \pi^*\sigma)$ of the dual Tonelli Hamiltonian $H : \mathrm{T}^*M \to \mathbb{R}$, $H(q, p) = \max\{pv - L(q, v) \mid v \in \mathrm{T}_q M\}$, see e.g. [Arn61, Nov82, AM16]. A particularly relevant special case of this setting is the

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electromagnetic one, when the Lagrangian L is of the form $L(q,v) = \frac{1}{2}g_q(v,v) - U(q)$ for some Riemannian metric g and some smooth potential $U : M \to \mathbb{R}$. In this situation, the system (L,σ) models the motion of a particle on M with kinetic and potential energies described by L and under the further effect of a Lorentz force described by σ. When the potential U vanishes, the dynamics of the system (L,σ) is a so-called magnetic geodesic flow.

In this paper, we will focus on the case where $M = S^2$. The energy function $E : \mathrm{T}M \to \mathbb{R}$, $E(q,v) := \partial_v L(q,v)v - L(q,v)$, is preserved along the motion. Therefore it is natural to study the dynamics of a magnetic Tonelli flow on a prescribed energy hypersurface $E^{-1}(e)$, and very different qualitative behaviors appear for different values of the energy e, see [CFP10] and references therein. For our purposes two energy values will bear special significance: $e_0(L)$ and $e_1(L,\sigma)$. The former is the minimal energy e such that the corresponding energy hypersurface $E^{-1}(e) \subset \mathrm{T}S^2$ projects onto the whole S^2. We postpone the precise definition of $e_1(L,\sigma)$ to Section 2. For now, we just mention that $e_1(L,\sigma) \geq e_0(L)$, and when σ is exact with primitive θ we have $e_1(L,d\theta) = c_u(L+\theta)$, where $c_u(L+\theta)$ is the Mañé critical value of the universal cover of $L+\theta$ (see e.g. [CI99, Abb13] for the definition of Mañé critical values).

The periodic orbits problem for magnetic geodesics was first studied by Novikov [Nov81, Nov82] in the early 1980s. The classical least action principle for the periodic orbits with prescribed energy is not directly available in this setting, due to the potential non-exactness of the magnetic 2-form. Novikov showed how to recover the variational principle in the universal cover of the space of periodic curves, and in his celebrated "throwing out cycles" method he proposed how to exploit the corresponding deck transformation in order to detect action values of periodic orbits (for the throwing out cycles method, see also [Tai83]). For magnetic geodesics on closed surfaces, waists were first studied by Taimanov in a series of papers [Tai91, Tai92a, Tai92b]. Taimanov's result is that, given a kinetic Lagrangian $L(q,v) = \frac{1}{2}g_q(v,v)$ and an oscillating magnetic 2-form σ on a closed 2-dimensional configuration space, there exists a waist α_e at the energy level e, for all $e \in (0, e_1(L,\sigma))$ (see also [CMP04] for a different proof). When σ is exact, Abbondandolo, Macarini, Mazzucchelli and Paternain [AMMP14] employed Taimanov's waist α_e on any energy level e belonging to a full measure subset of $(0, e_1(L,\sigma))$ in order to construct a sequence of minmax families giving an infinite number of (geometrically distinct) periodic orbits with energy e. Short afterwards, Asselle and Benedetti extended the result to non-exact σ on surfaces of genus at least one [AB15a, AB15b]. The results in [Tai91, Tai92a, Tai92b, AMMP14, AB15a, AB15b] have been further extended by Asselle and Mazzucchelli [AM16] to the general magnetic Tonelli setting. In this note we complete the picture by treating the last case remained open for the multiplicity problem: the 2-sphere. Namely, we are going to prove the following result.

Theorem 1.1. *Let $L : \mathrm{T}S^2 \to \mathbb{R}$ be a Tonelli Lagrangian, and σ a 2-form on S^2. For almost every $e \in (e_0(L), e_1(L,\sigma))$, the Lagrangian system of (L,σ) possesses infinitely many periodic orbits with energy e.*

We wish to stress that the existence of infinitely many periodic orbits on *all* energy values in $(e_0(L), e_1(L,\sigma))$ is still an open problem. In Theorem 1.1, as well as in [AMMP14, AB15a, AB15b], a negligible subset of energies must be excluded due to a lack of compactness in the variational setting that is employed. However, energy levels with only finitely many periodic orbits can be found above [Zil83, Ben16] as well as below [AM16] the interval $[e_0(L), e_1(L,\sigma)]$.

For closed surfaces M of genus at least one, any closed 2-form σ on M lifts to an exact 2-form on the universal cover of M. This allows to define the Mañé critical value of the universal cover $c_u(L,\sigma)$ for any Tonelli Lagrangian $L : \mathrm{T}M \to \mathbb{R}$. We set $e_1^*(L,\sigma) := \min\{e_1(L,\sigma), c_u(L,\sigma)\}$ if M has positive genus, and $e_1^*(L,\sigma) := e_1(L,\sigma)$ if $M = S^2$. The combination of Theorem 1.1 together with the above mentioned results in [AB15a, AB15b, AM16], yields the following statement about the multiplicity of periodic orbits on general closed surfaces.

Theorem 1.2. *Let M be a closed surface, $L : \mathrm{T}M \to \mathbb{R}$ a Tonelli Lagrangian, and σ a 2-form on M. For almost every $e \in (e_0(L), e_1^*(L,\sigma))$, the Lagrangian system of (L,σ) possesses infinitely many periodic orbits with energy e.* ■

Remark 1.3. The open interval $(e_0(L), e_1^*(L,\sigma))$ is not empty for instance if M is orientable, σ is oscillating, and the Lagrangian has the form of a kinetic energy $L(q,v) = \frac{1}{2} g_q(v,v)$ for some Riemannian metric g (see [AB15b]); in such case, $e_0(L) = 0$ is the minimal energy of the system. We recall that a 2-form σ on an orientable surface is oscillating when it satisfies $\sigma_{q_-} < 0$ and $\sigma_{q_+} > 0$ for some $q_-, q_+ \in M$. On non-orientable surfaces, any non-zero 2-form lifts to an oscillating 2-form on the orientation double cover. ■

This paper is dedicated to the memory of Abbas Bahri. Bahri was interested in the problem of periodic orbits of magnetic geodesic flows. In a joint work with Taimanov [BT98], he established the existence of periodic magnetic geodesics with prescribed energy on closed configuration spaces of arbitrary dimension under the assumption that the analog of the Ricci curvature for the Lagrangian system is positive.

The paper is organized as follows. In Section 2 we recall the variational setting for our periodic orbits problem: we provide the definition of the action 1-form η_e, and of its global primitive A_e on the universal cover of the space of loops; at the end we will review the definition of the energy values e_0 and e_1, and the notion of a waist for magnetic Tonelli systems. In Section 3 we provide the proof of Theorem 1.1.

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2. The primitive of the free-period action form

2.1. **The variational principle.** Let $L : \mathrm{T}S^2 \to \mathbb{R}$ be a Tonelli Lagrangian with associated energy function $E(q,v) = \partial_v L(q,v)v - L(q,v)$, and σ a 2-form on S^2. Since we will be interested in the Euler-Lagrange dynamics on a given energy hypersurface $E^{-1}(e)$, for some fixed $e \in \mathbb{R}$, we can modify the Tonelli Lagrangian far from $E^{-1}(e)$ and assume without loss of generality that each restriction $L|_{\mathrm{T}_q M}$ coincides with a polynomial of degree 2 outside a compact set. Let $\mathcal{M} := W^{1,2}(\mathbb{T}; S^2) \times (0, \infty)$, where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ is the 1-periodic circle. For each energy value $e \in \mathbb{R}$, we consider the free-period action 1-form η_e on $\mathcal{M}$ given by

$$\eta_e(\gamma, p)(\xi, q) = \mathrm{d}S_e(\gamma, p)(\xi, q) + \int_{\mathbb{T}} \sigma_{\gamma(t)}(\xi(t), \dot\gamma(t))\,\mathrm{d}t, \qquad (\gamma, p) \in \mathcal{M}, \quad (\xi, q) \in \mathrm{T}_{(\gamma,p)}\mathcal{M},$$

where $S_e : \mathcal{M} \to \mathbb{R}$ denotes the free-period action functional

$$S_e(\gamma, p) = p \int_{\mathbb{T}} L(\gamma(t), \dot\gamma(t)/p)\,\mathrm{d}t + p\,e.$$

By the least action principle, η_e vanishes at some $(\gamma, p) \in \mathcal{M}$ if and only if the p-periodic curve $\Gamma(t) := \gamma(t/p)$ is an orbit of the magnetic Tonelli system of (L, σ), see e.g. [AB16] and references therein.

The 1-form η_e is not exact if σ is not exact. In order to work with a primitive of η_e, following Novikov [Nov81, Nov82], we will lift it to the universal cover of $\mathcal{M}$. We see S^2 as the unit sphere in $\mathbb{R}^3$, oriented in the usual way, and we fix the point $x_0 = (-1, 0, 0) \in S^2$. We consider the universal cover

$$\pi : \widetilde{\mathcal{M}} \to \mathcal{M}.$$

As usual, we realize $\widetilde{\mathcal{M}}$ as the space of homotopy classes relative to the endpoints of continuous paths $u : [0, 1] \to \mathcal{M}$ starting at $u(0) = (x_0, 1)$. Here, we see x_0 as the constant loop at x_0. The projection map is given by $\pi([u]) = u(1)$. We have $\pi^*\eta_e = \mathrm{d}A_e$, where the functional

$$A_e : \widetilde{\mathcal{M}} \to \mathbb{R}$$

is defined as follows. Given $[u] \in \widetilde{\mathcal{M}}$, we write $u = (\gamma, p)$, where $\gamma(s) \in W^{1,2}(\mathbb{T}; S^2)$ and $p(s) \in (0, \infty)$ for all $s \in [0, 1]$. We see γ as a map of the form $\gamma : [0, 1] \times \mathbb{T} \to S^2$ by setting $\gamma(s, t) := \gamma(s)(t)$. We then set

$$A_e([u]) := S_e(u(1)) + \int_{[0,1]\times\mathbb{T}} \gamma^*\sigma.$$

Remark 2.1. Assume that $U \subsetneq S^2$ is a proper open subset, so that $\sigma|_U$ is exact with some primitive θ. Let $\mathcal{U} \subset \widetilde{\mathcal{M}}$ be a connected component of the open set of those $[u] \in \widetilde{\mathcal{M}}$ such that the periodic curve $u(1)$ is contained in U. Up to an additive constant, the restriction $A_e|_{\mathcal{U}}$ is equal to $S'_e \circ \pi|_{\mathcal{U}}$, where $S'_e : \mathcal{M} \to \mathbb{R}$ is the free-period action functional associated with the Lagrangian $L + \theta$, i.e.

$$S'_e(\gamma, p) = p \int_{\mathbb{T}} L(\gamma(t), \dot\gamma(t)/p)\,\mathrm{d}t + \int_\gamma \theta + p\,e. \qquad \blacksquare$$

It is well known that the fundamental group of the free loop space $W^{1,2}(\mathbb{T}; S^2)$ is isomorphic to $\mathbb{Z}$, and therefore so is the fundamental group of $\mathcal{M}$. A generator $[z]$ of $\pi_1(\mathcal{M}, (x_0, 1))$ can be defined as follows. For each $s \in \mathbb{T}$, consider the affine plane $\Sigma_s \subset \mathbb{R}^3$ orthogonal to the vector $(0, \cos(2\pi s), -\sin(2\pi s))$ and passing through x_0. We denote by $\zeta(s) \in W^{1,2}(\mathbb{T}; S^2)$ the closed curve with constant Euclidean speed whose support is precisely the intersection $\Sigma_s \cap S^2$, its starting point is $\zeta(s)(0) = x_0$,

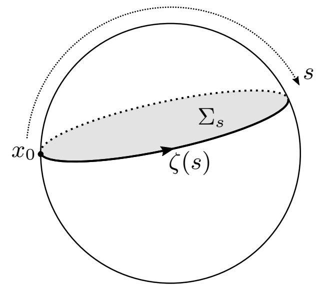

FIGURE 1. The map $\zeta : \mathbb{T} \to W^{1,2}(\mathbb{T}; S^2)$.

and, for all $s \neq 0$, its orientation is such that the ordered pair $\partial_s\zeta(s)(t), \partial_t\zeta(s)(t)$ agrees with the orientation of S^2, see Figure 1. We define $z := (\zeta, 1) : \mathbb{T} \to \mathcal{M}$. The group of deck transformations of the universal cover $\widetilde{\mathcal{M}}$ is generated by

$$Z : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}, \qquad Z([u]) = [z * u],$$

where $z * u(s) = z(2s)$ for all $s \in [0, 1/2]$, and $z * u(s) = u(2s-1)$ for all $s \in [1/2, 1]$. The action A_e varies under such a transformation as

$$A_e \circ Z([u]) = A_e([u]) + \int_{S^2} \sigma. \tag{2.1}$$

2.2. **Iterated curves.** For each $v = (\gamma, p) \in \mathcal{M}$, we denote by $v^m = (\gamma^m, mp) \in \mathcal{M}$ its m-fold iterate, where $\gamma^m(t) = \gamma(mt)$. The iteration map $\psi^m : \mathcal{M} \to \mathcal{M}$, $\psi^m(v) = v^m$, is smooth. We lift this map to a smooth map of the universal cover, so that the following diagram commutes

$$\begin{array}{ccc} \widetilde{\mathcal{M}} & \xrightarrow{\widetilde{\psi}^m} & \widetilde{\mathcal{M}} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{M} & \xrightarrow{\psi^m} & \mathcal{M} \end{array}$$

For instance, we can set $\widetilde{\psi}^m([u]) := [u^m]$, where

$$u^m(s) = \begin{cases} (x_0, 1 + 2s(m-1)) & \text{if } s \in [0, 1/2], \\ u(2s-1)^m & \text{if } s \in [1/2, 1]. \end{cases}$$

A remarkable property of the iteration map is given by the non-mountain pass Theorem for high iterates, which was first established for electromagnetic Lagrangians in [AMMP14, Theorem 2.6], and extended to general Tonelli Lagrangians in [AM16, Lemma 4.3 and proof of Theorem 1.2]. As we explained in Remark 2.1, A_e coincides locally with the free-period action functional of a suitable Tonelli Lagrangian, and therefore the non-mountain pass Theorem for high iterates holds for A_e as well.

Theorem 2.2 (Non-mountain pass Theorem for high iterates)**.** *Let $[v]$ be a critical point of A_e such that, for all $m \in \mathbb{N}$, the critical circle of $[v^m]$ is isolated in the set of critical points of A_e. There exists $m([v]) \in \mathbb{N}$ such that, for all integers $m > m([v])$, the following holds. There exists an (arbitrarily small) open neighborhood $\mathcal{W}$ of* *the critical circle of* $[v^m]$ *such that, if we set* $a := A_e([v^m])$*, the inclusion induces an injective map between path-connected components*

$$\pi_0(\{A_e < a\}) \hookrightarrow \pi_0(\{A_e < a\} \cup \mathcal{W}).$$ ■

2.3. **The critical values of the energy.** Let us single out two significant values of the energy. The first one is $e_0(L) := \max E(\cdot, 0)$, that is, the minimal energy e such that the corresponding energy hypersurface $E^{-1}(e)$ projects onto the whole S^2. The second value $e_1(L,\sigma) \geq e_0(L)$, which depends also on the magnetic form σ, is defined as the supremum of the energies $e \geq e_0(L)$ verifying the following condition: there exists a finite collection $(\gamma_1, p_1), ..., (\gamma_n, p_n) \in \mathcal{M}$ such that the γ_i's are smooth pairwise disjoint loops, $E(\gamma_i(\cdot), \dot\gamma_i(\cdot)/p_i) \equiv e$ for all $i = 1, ..., n$, the multicurve $\gamma_1 \cup ... \cup \gamma_n$ is the oriented boundary of a positively oriented compact embedded surface $\Sigma \subseteq S^2$, and we have

$$S_e(\gamma_1, p_1) + ... + S_e(\gamma_n, p_n) + \int_\Sigma \sigma < 0.$$

We recall that $e_1(L,\sigma)$ reduces to the classical Mañé critical value of $L + \theta$ in case σ is exact with primitive θ, see [AB15b].

The proof of Theorem 1.1 will build on the following existence result, which was originally proved by Taimanov [Tai91, Tai92b] in the case of electromagnetic Lagrangians (see also [CMP04] for an alternative proof), and further extended by Asselle and Mazzucchelli [AM16, Theorem 6.1] to the general case of magnetic Tonelli systems.

Theorem 2.3. *For every energy value* $e \in (e_0(L), e_1(L,\sigma))$*, the Lagrangian system of* (L,σ) *possesses a non-self-intersecting periodic orbit* (γ_e, p_e) *with energy* e *such that every element in* $\pi^{-1}(\gamma_e, p_e)$ *is a local minimizer of the action functional* A_e*.* ■

3. Proof of the Main Theorem

In this section we carry out the proof of Theorem 1.1. Since the case where the magnetic 2-form σ is exact is covered by [AMMP14], we focus on the case where σ is not exact, so that

$$\int_{S^2} \sigma \neq 0. \tag{3.1}$$

3.1. **Minimax procedures.** For each energy value $e \in (e_0(L), e_1(L,\sigma))$, consider the local minimizer (γ_e, p_e) of A_e given by Theorem 2.3, and choose an arbitrary $u_e \in \pi^{-1}(\gamma_e, p_e)$. We fix an arbitrary energy value

$$e_* \in (e_0(L), e_1(L,\sigma))$$

such that, for all $m \in \mathbb{N}$, the iterated critical point $[u_{e_*}^m]$ belongs to a critical circle that is isolated in $\mathrm{crit}(A_{e_*})$ (if there is no energy value e_* with such a property, there are infinitely many periodic orbits on every energy level in the range $(e_0(L), e_1(L,\sigma))$). The critical points $[u_{e_*}^m]$ are still local minimizers of A_{e_*}, as they are iterates of a local minimizer, see [AMP15, Lemma 3.1] and Remark 2.1.

Given any subset $Y \subset \widetilde{\mathcal{M}}$, for each $m \in \mathbb{N}$ we will write

$$Y^m := \widetilde\psi^m(Y) = \big\{[y^m] \bigm| [y] \in Y\big\}.$$

The Palais-Smale condition holds locally for the free-period action functional of Tonelli Lagrangians, see [Con06, Prop. 3.12] or [Abb13, Lemma 5.3]. This, together with Remark 2.1, implies that the functional A_{e_*} satisfies the Palais-Smale condition locally as well. Therefore, a sufficiently small bounded open neighborhood $\mathcal{W}$ of the critical circle of $[u_{e_*}]$ does not contain other critical circles of A_{e_*} and satisfies

$$\inf_{\partial\mathcal{W}} A_{e_*} > A_{e_*}([u_{e_*}]),$$
$$\mathcal{W}^{m_0} \cap Z^{n_1}(\mathcal{W}^{m_1}) = \varnothing \quad \text{whenever} \quad (m_0, 0) \neq (m_1, n_1).$$

For any $e \in (e_0(L), e_1(L,\sigma))$, we denote by M_e the closure of the set of local minimizers of $A_e|_{\mathcal{W}}$. For all $m_0, m_1 \in \mathbb{N}$ and $n_0, n_1 \in \mathbb{Z}$ such that $(m_0, n_0) \neq (m_1, n_1)$, we denote by

$$\mathcal{P}_e(m_0, n_0, m_1, n_1)$$

the family of continuous paths $\Theta : [0,1] \to \widetilde{\mathcal{M}}$ such that $\Theta(0) \in Z^{n_0}(M_e^{m_0})$ and $\Theta(1) \in Z^{n_1}(M_e^{m_1})$. We define the corresponding minmax value

$$c_e(m_0, n_0, m_1, n_1) := \inf \big\{ \max A_e \circ \Theta \ \big|\ \Theta \in \mathcal{P}_e(m_0, n_0, m_1, n_1) \big\}.$$

Lemma 3.1. *There is an open neighborhood $I \subset (e_0(L), e_1(L,\sigma))$ of e_* such that*

(i) *M_e is a non-empty compact set for all $e \in I$,*
(ii) *for each $e, e' \in I$, we have $\max A_{e'}|_{M_{e'}} < \inf A_e|_{\partial\mathcal{W}}$,*
(iii) *for each $m_0, m_1 \in \mathbb{N}$ and $n_0, n_1 \in \mathbb{Z}$, the function $e \mapsto c_e(m_0, n_0, m_1, n_1)$ is well defined and monotone increasing in I.*

Proof. The proof is entirely analogous to the arguments in [AMMP14, Lemmas 3.1–3.3] and it will be omitted. ∎

3.2. **The valley of short curves with low period.** We equip our sphere S^2 with an arbitrary Riemannian metric g, and $\mathcal{M}$ with the Riemannian metric

$$\langle (\xi, r), (\eta, s) \rangle = \int_{\mathbb{T}} \Big(g(\xi, \eta) + g(D_t\xi, D_t\eta) \Big)\, \mathrm{d}t + rs, \qquad \forall (\xi, r), (\eta, s) \in \mathrm{T}_{(\gamma,p)}\mathcal{M}, \tag{3.2}$$

where D_t denotes the covariant derivative associated to g. The space $\mathcal{M}$ is not complete with respect to the Riemannian metric (3.2), nor is its universal cover equipped with the pulled-back Riemannian metric. Indeed, there are Cauchy sequences $\{(\gamma_n, p_n) \mid n \in \mathbb{N}\} \subset \mathcal{M}$ such that $p_n \to 0$. However, it turns out that this does not pose any problem while applying arguments from non-linear analysis to the functional A_e. Indeed, the functional A_e has a "valley" near the non-complete ends of $\mathcal{M}$, as we will review now (see [Con06, Section 3] and [AB16, Section 3] for analogous arguments in slightly different settings).

We write $\|\dot\gamma\|_{L^2}$ for the L^2-norm of the derivative of any curve $\gamma \in W^{1,2}(\mathbb{T}; S^2)$ measured with respect to g, i.e.

$$\|\dot\gamma\|_{L^2}^2 = \int_{\mathbb{T}} g(\dot\gamma(t), \dot\gamma(t))\, \mathrm{d}t.$$

We introduce the open subsets

$$\mathcal{U}_\tau := \big\{ (\gamma, p) \in \mathcal{M} \ \big|\ \|\dot\gamma\|_{L^2}^2 < \tau\, p, \quad p < \tau \big\}, \qquad \tau > 0. \tag{3.3}$$

If τ is small enough, $\mathcal{U}_\tau$ is connected and evenly covered by $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$. Namely, there exists a connected component $\mathcal{V}_\tau \subset \pi^{-1}(\mathcal{U}_\tau)$ such that $\pi^{-1}(\mathcal{U}_\tau)$ can be written as a disjoint union

$$\pi^{-1}(\mathcal{U}_\tau) = \bigsqcup_{n\in\mathbb{Z}} Z^n(\mathcal{V}_\tau).$$

We choose such a connected component $\mathcal{V}_\tau$ so that, for all $[u] = [(\gamma, p)] \in \mathcal{V}_\tau$ with $\gamma(1)$ stationary curve at some point $q \in S^2$, we have

$$A_e([u]) = p(1)\big(L(q, 0) + e\big).$$

Lemma 3.2. *For all $\tau > 0$ sufficiently small, we have*

$$\inf A_e|_{\mathcal{V}_\tau} = 0, \qquad \inf A_e|_{\partial\mathcal{V}_\tau} > 0.$$

Moreover

$$\lim_{\tau\to 0^+} \big(\sup A_e|_{\mathcal{V}_\tau}\big) = 0.$$

Proof. We cover the sphere with two open balls $D_1, D_2 \subset S^2$, and choose a primitive θ_i of σ on D_i. Let $\tau > 0$ be sufficiently small so that for any $\gamma \in W^{1,2}(\mathbb{T}; S^2)$ with length less than τ there exists $\iota(\gamma) \in \{1, 2\}$ such that γ is entirely contained in $D_{\iota(\gamma)}$. The restriction of the functional A_e to $\mathcal{V}_\tau$ takes the following form: for each $[u] \in \mathcal{V}_\tau$ with $(\gamma, p) := \pi([u])$, we have

$$A_e([u]) = p \int_{\mathbb{T}} L(\gamma(t), \dot\gamma(t)/p)\,\mathrm{d}t + p\,e + \int_\gamma \theta_{\iota(\gamma)}.$$

Since we are assuming that the restriction of the Tonelli Lagrangian L to any fiber of $\mathrm{T}M$ is a polynomial of degree 2 outside a compact set, there exist constants $0 < h_1 < h_2$ such that, for all $(q, v) \in \mathrm{T}M$, we have

$$\begin{aligned} L(q, v) &\geq L(q, 0) + \partial_v L(q, 0)v + h_1\, g_q(v, v) \\ &\geq -E(q, 0) + \partial_v L(q, 0)v + h_1\, g_q(v, v) \\ &\geq -e_0(L) + \partial_v L(q, 0)v + h_1\, g_q(v, v), \end{aligned} \tag{3.4}$$

and

$$L(q, v) \leq h_2\big(g_q(v, v) + 1\big). \tag{3.5}$$

We denote by λ the 1-form on S^2 given by $\partial_v L(\cdot, 0)$. The lower bound (3.4) implies that, for all $[u] \in \mathcal{V}_\tau$ with $(\gamma, p) := \pi([u])$, we have

$$\begin{aligned} A_e([u]) &\geq h_1 \frac{\|\dot\gamma\|^2_{L^2}}{p} + \underbrace{\big(e - e_0(L)\big)}_{>0}\, p - \left| \int_\gamma (\lambda + \theta_{\iota(\gamma)}) \right| \\ &\geq h_1 \frac{\|\dot\gamma\|^2_{L^2}}{p} + \big(e - e_0(L)\big)\, p - \frac{1}{4} \|\mathrm{d}\lambda + \underbrace{\mathrm{d}\theta_{\iota(\gamma)}}_{\sigma}\|_{L^\infty}\, \|\dot\gamma\|^2_{L^2} \end{aligned}$$

where the latter inequality follows from [Abb13, Lemma 7.1]. This readily implies that $A_e > 0$ on $\mathcal{V}_\tau$ provided

$$\frac{h_1}{\tau} > \frac{1}{4}\|\mathrm{d}\lambda + \sigma\|_{L^\infty}.$$

Assume now that $[u] \in \partial\mathcal{V}_\tau$. If $p = \tau$, we have

$$A_e([u]) > (e - e_0(L))\,\tau > 0.$$

If $p < \tau$, then $\|\dot\gamma\|^2_{L^2} = p\,\tau$, and therefore

$$A_e([u]) \geq h_1\tau - \frac{1}{4}\|\mathrm{d}\lambda + \sigma\|_{L^\infty}\,\tau^2 > 0.$$

Overall, this proves that $\inf A_e|_{\partial\mathcal{V}_\tau} > 0$.

Inequality (3.5) implies that, for all $[u] \in \mathcal{V}_\tau$ with $(\gamma, p) := \pi([u])$, we have

$$\begin{aligned} A_e([u]) &\leq h_2 \frac{\|\dot\gamma\|^2_{L^2}}{p} + h_2\,p + e\,p + \int_\gamma \theta_{\iota(\gamma)} \\ &\leq h_2 \frac{\|\dot\gamma\|^2_{L^2}}{p} + h_2\,p + e\,p + \frac{1}{4}\|\sigma\|_{L^\infty}\,\|\dot\gamma\|^2_{L^2} \\ &\leq h_2\,\tau + h_2\,\tau + e\,\tau + \frac{1}{4}\|\sigma\|_{L^\infty}\,\tau^2, \end{aligned}$$

where, as before, the second inequality follows from [Abb13, Lemma 7.1]. This readily implies that $\sup A_e|_{\mathcal{V}_\tau} \to 0$ as $\tau \to 0^+$, which, together with the fact that $A_e > 0$ on $\mathcal{V}_\tau$, also implies that $\inf A_e|_{\mathcal{V}_\tau} = 0$. ■

3.3. **Essential families.** Let us fix an energy value $e \in I$. We say that a union of critical circles

$$\mathcal{E} \subset \mathrm{crit} A_e \cap \{A_e = c_e(m_0, n_0, m_1, n_1)\}$$

is an **essential family** for $\mathcal{P}_e(m_0, n_0, m_1, n_1)$ when for every neighborhood $\mathcal{U}$ of $\mathcal{E}$ there exists a path $\Theta \in \mathcal{P}_e(m_0, n_0, m_1, n_1)$ whose image $\Theta([0,1])$ is contained in the union $\mathcal{U} \cup \{A_e < c_e(m_0, n_0, m_1, n_1)\}$.

We denote by I_{discr} the subset of those $e \in (e_0(L), e_1(L,\sigma))$ such that the set of critical points $\mathrm{crit}(A_e)$ is a union of isolated critical circles (that is, the periodic orbits with energy e are isolated). Notice that every energy level $e \in (e_0(L), e_1(L,\sigma)) \setminus I_{\mathrm{discr}}$ contains infinitely many periodic orbits. The existence of essential families can be guaranteed on generic energy levels in I_{discr}. The precise statement is the following.

Lemma 3.3. *There is a subset $I' \subseteq I$ of full Lebesgue measure such that, for all $e \in I' \cap I_{\mathrm{discr}}$, $m_0, m_1 \in \mathbb{N}$, and $n_0, n_1 \in \mathbb{Z}$ with $(m_0, n_0) \neq (m_1, n_1)$, the space of paths $\mathcal{P}_e(m_0, n_0, m_1, n_1)$ admits an essential family.*

Proof. The proof goes along the lines of the one of [AMMP14, Lemma 3.5], but the fact that we are working on the universal cover of $\mathcal{M}$ with the functional A_e requires some variations of the original argument, and therefore we provide full details for the reader's convenience.

For all $m_0, m_1 \in \mathbb{N}$ and $n_0, n_1 \in \mathbb{Z}$ such that $(m_0, n_0) \neq (m_1, n_1)$, we denote by $I(m_0, n_0, m_1, n_1)$ the subset of those $e' \in I$ such that the function

$$e \mapsto c_e(m_0, n_0, m_1, n_1) \tag{3.6}$$

is differentiable at e'. By Lemma 3.1(iii), the function (3.6) is monotone increasing in e, and therefore $I(m_0, n_0, m_1, n_1)$ is a full measure subset of I. We define the subset I' of the statement as

$$I' := \bigcap_{(m_0,n_0)\neq(m_1,n_1)} I(m_0,n_0,m_1,n_1).$$

Being a countable intersection of full Lebesgue measure subsets of I, the subset $I' \subset I$ has full Lebesgue measure as well.

Now, we fix $e \in I' \cap I_{\mathrm{discr}}$ and two distinct $(m_0,n_0),(m_1,n_1) \in \mathbb{N}\times\mathbb{Z}$. In order to simplify the notation, we will just write c_e and $\mathcal{P}_e$ for $c_e(m_0,n_0,m_1,n_1)$ and $\mathcal{P}_e(m_0,n_0,m_1,n_1)$ respectively. We choose an arbitrary strictly decreasing sequence $\{e_\alpha \mid \alpha\in\mathbb{N}\} \subset I$ such that $e_\alpha \to e$ as $\alpha\to\infty$, and we set $\epsilon_\alpha := e_\alpha - e$. By definition of I', there exists $k_0 = k_0(e) > 0$ such that

$$|c_{e_\alpha} - c_e| \leq k_0 \epsilon_\alpha, \qquad \forall \alpha \in \mathbb{N}.$$

For all $[u] = [(\gamma,p)] \in \widetilde{\mathcal{M}}$ such that $A_e([u]) \geq c_e - \epsilon_\alpha$ and $A_{e_\alpha}([u]) \leq c_{e_\alpha} + \epsilon_\alpha$, the period $p(1)$ of the curve $u(1) \in \mathcal{M}$ can be bounded as

$$p(1) = \frac{A_{e_\alpha}([u]) - A_e([u])}{\epsilon_\alpha} \leq \frac{c_{e_\alpha} + \epsilon_\alpha - c_e + \epsilon_\alpha}{\epsilon_\alpha} \leq k_0 + 2 =: k_1,$$

while the action $A_e([u])$ can be bounded as

$$A_e([u]) \leq A_{e_\alpha}([u]) \leq c_{e_\alpha} + \epsilon_\alpha \leq c_e + (k_0+1)\epsilon_\alpha \leq c_e + k_1\epsilon_\alpha.$$

We introduce the subspaces

$$\mathcal{X}_r := \Big\{ [u] = [(\gamma,p)] \in \widetilde{\mathcal{M}} \ \Big|\ p(1) \leq r \Big\}, \qquad r > 0.$$

By the definition of the minmax value c_{e_α} and by the estimates that we have just provided, for each $\alpha \in \mathbb{N}$ there exists a path $\Theta_\alpha \in \mathcal{P}_{e_\alpha}$ such that

$$\Theta_\alpha([0,1]) \subset \{A_e \leq c_e - \epsilon_\alpha\} \cup \big(\mathcal{X}_{k_1} \cap \{A_e \leq c_e + k_1\epsilon_\alpha\}\big).$$

We recall that, by the definition of the spaces of paths $\mathcal{P}_{e_\alpha}$, we have that

$$\Theta_\alpha(i) \in Z^{n_i}(M^{m_i}_{e_\alpha}) \subset Z^{n_i}(\mathcal{W}^{m_i}), \qquad i = 0,1.$$

Lemma 3.1(ii) readily implies that we can attach two suitable tails to the path Θ_α: we can find two continuous paths

$$\begin{aligned} \Phi_\alpha :&[0,1] \to Z^{n_0}(\mathcal{W}^{m_0}) \cap \{A_e \leq A_e(\Theta_\alpha(0))\},\\ \Psi_\alpha :&[0,1] \to Z^{n_1}(\mathcal{W}^{m_1}) \cap \{A_e \leq A_e(\Theta_\alpha(1))\}, \end{aligned}$$

such that $\Phi_\alpha(0) \in Z^{n_0}(M^{m_0}_e)$, $\Phi_\alpha(1) = \Theta_\alpha(0)$, $\Psi_\alpha(0) = \Theta_\alpha(1)$, and $\Psi_\alpha(1) \in Z^{n_1}(M^{m_1}_e)$; see [AMMP14, Lemma 3.2] for a proof of this elementary fact. Since the open set $\mathcal{W}$ is bounded, there exists $k_2 > k_1$ large enough such that

$$Z^{n_0}(\mathcal{W}^{m_0}) \cup Z^{n_1}(\mathcal{W}^{m_1}) \subset \mathcal{X}_{k_2}.$$

We define the continuous path

$$\Upsilon_\alpha : [0,1] \to \{A_e \leq c_e - \epsilon_\alpha\} \cup \big(\mathcal{X}_{k_2} \cap \{A_e \leq c_e + k_1\epsilon_\alpha\}\big),$$

$$\Upsilon_\alpha(s) := \begin{cases} \Phi_\alpha(3s), & s \in [0,1/3],\\ \Theta_\alpha(3(s-1/3)), & s \in [1/3,2/3],\\ \Psi_\alpha(3(s-2/3)), & s \in [2/3,1]. \end{cases}$$

Notice that $\Upsilon_\alpha \in \mathcal{P}_e$, and $\max A_e \circ \Upsilon_\alpha \to c_e$ as $\alpha \to \infty$.

We claim that $\mathrm{crit}(A_e) \cap A_e^{-1}(c_e) \cap \mathcal{X}_{k_2+2}$ is an essential family for $\mathcal{P}_e$. Let $\mathcal{U} \subset \widetilde{\mathcal{M}}$ be an arbitrary open set such that

$$\mathcal{U} \cap \mathrm{crit}(A_e) = \mathrm{crit}(A_e) \cap A_e^{-1}(c_e) \cap \mathcal{X}_{k_2+2}.$$

Our goal for the remaining of the proof is to deform one of our paths Υ_α, away from its endpoints, so that the modified path will have image inside $\{A_e < c_e\} \cup \mathcal{U}$. Notice that, since $e \in I_{\mathrm{discr}}$, if $\mu > 0$ is small enough we have

$$\mathcal{U} \cap \mathrm{crit}(A_e) = \mathrm{crit}(A_e) \cap A_e^{-1}[c_e - \mu, c_e + \mu] \cap \mathcal{X}_{k_2+2}, \tag{3.7}$$

and $\mathcal{U}$ contains at most finitely many critical circles of A_e. In particular, we can find a smaller open neighborhood $\mathcal{U}' \subset \mathcal{U}$ of $\mathcal{U} \cap \mathrm{crit}(A_e)$ and some $\ell > 0$ such that every smooth path $\Theta : [0,1] \to \overline{\mathcal{U}}$ with $\Theta(0) \in \mathcal{U}'$ and $\Theta(1) \in \partial\mathcal{U}$ has length at least ℓ. Here, the length is the one measured with respect to the pull-back of the Riemannian metric (3.2) to the universal cover $\widetilde{\mathcal{M}}$.

Consider the open subsets $\mathcal{U}_\tau \subset \mathcal{M}$ introduced in (3.3), and the selected connected components of their preimage $\mathcal{V}_\tau \subset \pi^{-1}(\mathcal{U}_\tau)$. Since $e \in I_{\mathrm{discr}}$, the set M_e is the union of finitely many critical circles of A_e. In particular, there exists $\tau_2 > 0$ small enough such that

$$\{\Upsilon_\alpha(0), \Upsilon_\alpha(1) \mid \alpha \in \mathbb{N}\} \cap \pi^{-1}(\mathcal{U}_{\tau_2}) = \varnothing.$$

If needed, we reduce $\tau_2 > 0$ so that the open subset $\mathcal{U}_{\tau_2}$ is connected and evenly covered by $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$. By Lemma 3.2, there exist $\delta > 0$ and $0 < \tau_1 < \tau_2$ such that, for all $n \in \mathbb{N}$,

$$\inf A_e|_{\partial(Z^n(\mathcal{V}_{\tau_2}))} - \sup A_e|_{Z^n(\mathcal{V}_{\tau_1})} \geq \delta. \tag{3.8}$$

Finally, we fix an index $\alpha \in \mathbb{N}$ large enough so that

$$k_1 \epsilon_\alpha < \min\big\{\mu, \delta\big\}. \tag{3.9}$$

In the following, we will denote by $\|\cdot\|$ the Riemannian norm induced by the Riemannian metric (3.2). With a slight abuse of notation, we will denote by $\|\cdot\|$ also the Riemannian norm that is pulled-back to the universal cover $\widetilde{\mathcal{M}}$. Fix $\tau_0 \in (0, \tau_1)$ and introduce a vector field on $\widetilde{\mathcal{M}}$ of the form $V := f\,\nabla A_e$, for some suitable smooth function $f : \widetilde{\mathcal{M}} \to [-1, 0]$, such that

(i) $\|V([u])\| \leq 2$ for all $[u] \in \widetilde{\mathcal{M}}$,
(ii) $\mathrm{supp}(V) \subset A_e^{-1}[c_e - \epsilon_{\alpha-1}, c_e + k_1\epsilon_{\alpha-1}] \setminus \pi^{-1}(\mathcal{U}_{\tau_0})$,
(iii) $\mathrm{d}A_e([u])V([u]) \leq -\min\big\{\|\nabla A_e([u])\|^2, 1\big\}$ for all $[u] \in \widetilde{\mathcal{M}} \setminus \pi^{-1}(\mathcal{U}_{\tau_1})$ such that $A_e([u]) \in [c_e - \epsilon_\alpha, c_e + k_1\epsilon_\alpha]$.

We denote by $\phi_t : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ the flow of V. This flow is complete. Indeed, since the vector field V is uniformly bounded, the flow lines that may not be defined for all positive time are those that enter all sets $\mathcal{X}_r$, for $r > 0$ arbitrarily small. Since V is non-negatively proportional to $-\nabla A_e$, its flow lines are non-negative reparametrizations of those of $-\nabla A_e$. Finally, if a flow line of $-\nabla A_e$ is not defined for all positive times, then it must enter the set $\pi^{-1}(\mathcal{U}_{\tau_0})$ (see [AB16, Proposition 3.1(2)] for a proof of this fact), but this latter set is outside the support of V. Actually, since $\|V\| \leq 2$, we have

$$\phi_1(\mathcal{X}_{k_2}) \subset \mathcal{X}_{k_2+2}.$$

The free-period action form η_e satisfies a generalized Palais-Smale condition on subsets of $\mathcal{M}$ where the period is bounded from above and bounded away from zero, see [AB16, Theorem 2.1(2)]. Moreover, for each sequence $\{(\gamma_n, p_n) \mid n \in \mathbb{N}\} \subset \mathcal{M}$ such that $p_n \to 0$ and $\|\eta_e(\gamma_n, p_n)\| \to 0$ as $n \to \infty$, we have $\|\dot\gamma_n\|^2_{L^2}/p_n \to 0$ as $n \to \infty$, see [AB16, Theorem 2.1(1)]. In particular, (γ_n, p_n) belongs to $\mathcal{U}_{\tau_0}$ for n large enough. This, together with (3.7), implies that there exists a constant $\nu \in (0,1)$ such that

$$\|\nabla A_e([u])\| \geq \nu, \quad \forall [u] \in A_e^{-1}[c_e - \mu, c_e + \mu] \cap \mathcal{X}_{k_2+2} \setminus (\pi^{-1}(\mathcal{U}_{\tau_0}) \cup \mathcal{U}'). \tag{3.10}$$

We fix an index $\beta \geq \alpha$ large enough so that $k_1 \epsilon_\beta < \min\{\ell\nu, \nu^2\}$, which together with (3.9) implies

$$k_1 \epsilon_\beta < \min\{\mu, \delta, \ell\nu, \nu^2\}. \tag{3.11}$$

The composition $\phi_1 \circ \Upsilon_\beta$ belongs to $\mathcal{P}_e$. We claim that its image $\phi_1 \circ \Upsilon_\beta([0,1])$ is contained in $\{A_e < c_e\} \cup \mathcal{U}$, which sets our goal for the proof. First of all, since A_e does not increase along the flow lines of ϕ_t, we have

$$A_e(\phi_t \circ \Upsilon_\beta) \leq A_e(\Upsilon_\beta) \leq c_e + k_1\epsilon_\beta, \qquad \forall t \in [0,1]. \tag{3.12}$$

There are three possible cases to consider:

- If $\phi_t \circ \Upsilon_\beta(s) \in \mathcal{U}'$ for some $t \in [0,1]$ and $\phi_1 \circ \Upsilon_\beta(s) \notin \mathcal{U}$, Equations (3.10), (3.11), and (3.12) imply that
$$\begin{aligned} A_e(\phi_1 \circ \Upsilon_\beta(s)) &= A_e(\phi_t \circ \Upsilon_\beta(s)) + \int_t^1 \mathrm{d}A_e(\phi_r \circ \Upsilon_\beta(s)) V(\phi_r \circ \Upsilon_\beta(s))\, \mathrm{d}r \\ &\leq c_e + k_1\epsilon_\beta - \nu \int_t^1 \|V(\phi_r \circ \Upsilon_\beta(s))\|\, \mathrm{d}r \\ &\leq c_e + k_1\epsilon_\beta - \ell\nu \\ &< c_e. \end{aligned}$$
- If $\phi_t \circ \Upsilon_\beta(s) \in \pi^{-1}(\mathcal{U}_{\tau_1})$ for some $t \in [0,1]$, then, since $\phi_t \circ \Upsilon_\beta(s) \notin \pi^{-1}(\mathcal{U}_{\tau_2})$, Equations (3.8), (3.11), and (3.12) imply
$$A_e(\phi_1 \circ \Upsilon_\beta(s)) \leq A_e(\phi_t \circ \Upsilon_\beta(s)) \leq c_e + k_1\epsilon_\beta - \delta < c_e.$$
- If $\phi_t \circ \Upsilon_\beta(s) \notin \mathcal{U}' \cup \pi^{-1}(\mathcal{U}_{\tau_1})$ for all $t \in [0,1]$, then property (iii) in the definition of V above, together with Equations (3.10), (3.11), and (3.12), implies
$$\begin{aligned} A_e(\phi_1 \circ \Upsilon_\beta(s)) &= A_e(\Upsilon_\beta(s)) + \int_0^1 \mathrm{d}A_e(\phi_r \circ \Upsilon_\beta(s)) V(\phi_r \circ \Upsilon_\beta(s))\, \mathrm{d}r \\ &= c_e + k_1\epsilon_\beta - \int_0^1 \|\mathrm{d}A_e(\phi_r \circ \Upsilon_\beta(s))\|^2\, \mathrm{d}r \\ &\leq c_e + k_1\epsilon_\beta - \nu^2 \\ &< c_e. \end{aligned}$$

Overall, we showed that, for an arbitrary $s \in [0,1]$, if $\phi_1 \circ \Upsilon_\beta(s)$ is not contained in $\mathcal{U}$, then it is contained in the sublevel set $\{A_e < c_e\}$. ■

Lemma 3.4. *For each $e \in I' \cap I_{\mathrm{discr}}$ and $[v] \in \mathrm{crit}(A_e)$, there exists a constant $m([v]) \in \mathbb{N}$ with the following property. Consider the critical circle $\mathcal{C}$ of a critical point $Z^n([v^m])$, where $n \in \mathbb{Z}$ and $m > m([v])$. If $\mathcal{E}$ is an essential family containing $\mathcal{C}$, then $\mathcal{E} \setminus \mathcal{C}$ is an essential family for the same space of paths as well.*

Proof. We set $a_{m,n} := A_e(Z^n([v^m]))$, where $m \in \mathbb{N}$ and $n \in \mathbb{Z}$. By Theorem 2.2, there exists $m([v]) \in \mathbb{N}$ such that, for all integers $m > m([v])$, the following statement holds. There exists an (arbitrarily small) open neighborhood $\mathcal{W}$ of the critical circle of $[v^m]$ such that the inclusion induces an injective map between path-connected components

$$\pi_0\big(\{A_e < a_{m,0}\}\big) \hookrightarrow \pi_0\big(\{A_e < a_{m,0}\} \cup \mathcal{W}\big).$$

For every $n \in \mathbb{Z}$, we denote by $\mathcal{W}_n := Z^n(\mathcal{W})$ the corresponding neighborhood of the critical circle $\mathcal{C}$ of $Z^n([v^m])$. Clearly, the inclusion induces an injective map

$$\pi_0\big(\{A_e < a_{m,n}\}\big) \hookrightarrow \pi_0\big(\{A_e < a_{m,n}\} \cup \mathcal{W}_n\big). \tag{3.13}$$

Now, assume that $\mathcal{C}$ belongs to an essential family $\mathcal{E}$ for $\mathcal{P}_e(m_0, n_0, m_1, n_1)$. In particular $a_{m,n} = c_e(m_0, n_0, m_1, n_1)$.

We require the neighborhood $\mathcal{W}$ to be small enough so that for all neighborhoods $\mathcal{W}'$ of $\mathcal{E}\setminus\mathcal{C}$ sufficiently small, we have $\overline{\mathcal{W}_n}\cap\overline{\mathcal{W}'} = \varnothing$. The existence of such a disjoint $\overline{\mathcal{W}'}$ is guaranteed by the fact that the set of critical points of A_e comes in isolated critical circles. Since, by Lemma 3.3, $\mathcal{P}_e(m_0, n_0, m_1, n_1)$ admits an essential family, there exists a continuous path $\Theta \in \mathcal{P}_e(m_0, n_0, m_1, n_1)$ whose image is contained in the union

$$\mathcal{W}_n \cup \mathcal{W}' \cup \{A_e < c_e(m_0, n_0, m_1, n_1)\}.$$

Notice that

$$\max\big\{A_e(\Theta(0)), A_e(\Theta(1))\big\} < c_e(m_0, n_0, m_1, n_1). \tag{3.14}$$

Indeed, $\Theta(0)$ and $\Theta(1)$ belong to distinct critical circles that are isolated local minimizers of A_e, and this latter functional satisfies the Palais-Smale condition locally.

By (3.14) and since the map (3.13) is injective, there exists another path $\Theta' \in \mathcal{P}_e(m_0, n_0, m_1, n_1)$ whose image is contained in the union

$$\mathcal{W}' \cup \{A_e < c_e(m_0, n_0, m_1, n_1)\}.$$

Therefore, $\mathcal{E} \setminus \mathcal{C}$ is also an essential family for $\mathcal{P}_e(m_0, n_0, m_1, n_1)$. ∎

Now, let I_{finite} be the (possibly empty) subset of those energy values $e \in I_{\text{discr}}$ such that there are only finitely many (non-iterated) periodic orbits with energy e. In order to prove Theorem 1.1, all we need to do is to prove that the intersection $I' \cap I_{\text{finite}}$ is empty. We will show this in Theorem 3.7, after exploring what would happen on energy values in $I' \cap I_{\text{finite}}$.

Lemma 3.5. *For each energy level $e \in I' \cap I_{\text{finite}}$ and compact interval $[a_0, a_1] \subset \mathbb{R}$, there exists a finite union of critical circles $\mathcal{E} \subset \text{crit}(A_e)$ such that, for all $m_0, m_1 \in \mathbb{N}$ and $n_0, n_1 \in \mathbb{Z}$ with $c_e(m_0, n_0, m_1, n_1) \in [a_0, a_1]$, $\mathcal{E}$ contains an essential family for $\mathcal{P}_e(m_0, n_0, m_1, n_1)$.*

Proof. Let $(\gamma_1, p_1), ..., (\gamma_r, p_r)$ be the only non-iterated periodic orbits with energy e, where r is some natural number, and choose $[v_i] \in \pi^{-1}(\gamma_i, p_i)$ for all $i = 1, ..., r$. Consider the constants $m([v_i]) \in \mathbb{N}$ given by Lemma 3.4, so that if we remove the critical circle of any $Z^n([v_i^m])$ with $n \in \mathbb{Z}$ and $m > m_{\max}$ from an essential family contained in $A_e^{-1}(c_e(m_0,n_0,m_1,n_1))$, the result is still an essential family for the same space of paths. We set

$$m_{\max} := \max\big\{m([v_1]),...,m([v_r])\big\} \in \mathbb{N}$$

By Equations (2.1) and (3.1), we infer that there exists $n_{\max} \in \mathbb{N}$ such that $A_e(Z^n([v_i^m])) \not\in [a_0,a_1]$ for all $i \in \{1,...,r\}$, $m \in \mathbb{N}$, and $n \in \mathbb{Z}$ with $m \leq m_{\max}$ and $|n| > n_{\max}$. We claim that the statement of the lemma holds taking

$$\mathcal{E} := \Big\{ Z^n([v_i^m]) \ \Big|\ i \in \{1,...,r\},\ 1 \leq m \leq m_{\max},\ |n| \leq n_{\max} \Big\}.$$

Indeed, consider $m_0, m_1 \in \mathbb{N}$ and $n_0, n_1 \in \mathbb{Z}$ such that $c_e(m_0,n_0,m_1,n_1) \in [a_0,a_1]$. Let $\mathcal{E}'$ be an essential family for $\mathcal{P}_e(m_0,n_0,m_1,n_1)$, whose existence is guaranteed by Lemma 3.3. By Lemma 3.4, if we remove from $\mathcal{E}'$ all the critical circles of periodic orbits of the form $Z^n([v_i^m])$ for $m > m_{\max}$, the resulting set is still an essential family for $\mathcal{P}_e(m_0,n_0,m_1,n_1)$. Therefore, $\mathcal{E}' \cap \mathcal{E}$ is an essential family for $\mathcal{P}_e(m_0,n_0,m_1,n_1)$. ∎

Let $m_0 \in \mathbb{N}$ and $n_0 \in \mathbb{Z}$. For all $m_1 \in \mathbb{N}$, and $n_1 \in \mathbb{Z}$ we know that $c_e(m_0,n_0,m_1,n_1)$ is bounded from below by $\min A_e|_{Z^{n_0}(M_e^{m_0})}$. Hence, the following quantity is a well-defined real number:

$$c_e(m_0,n_0) := \inf \left\{ c_e(m_0,n_0,m_1,n_1) \ \middle|\ \begin{array}{l} m_1 \in \mathbb{N},\ n_1 \in \mathbb{Z} \\ \text{with } (m_1,n_1) \neq (m_0,n_0) \end{array} \right\}.$$

Notice that the deck transformation Z^k induces a homeomorphism between the spaces of paths $\mathcal{P}_e(m_0,n_0,m_1,n_1)$ and $\mathcal{P}_e(m_0,n_0+k,m_1,n_1+k)$, and we have

$$c_e(m_0,n_0+k,m_1,n_1+k) = c_e(m_0,n_0,m_1,n_1) + k\int_{S^2}\sigma, \qquad \forall k \in \mathbb{Z}.$$

This readily implies

$$c_e(m_0,n_0+k) = c_e(m_0,n_0) + k\int_{S^2}\sigma, \qquad \forall k \in \mathbb{Z}. \tag{3.15}$$

The infimum in the definition of $c_e(m_0,n_0)$ is actually attained provided $e \in I' \cap I_{\mathrm{finite}}$.

Lemma 3.6. *If $e \in I' \cap I_{\mathrm{finite}}$, for all $(m_0,n_0) \in \mathbb{N}\times\mathbb{Z}$ there exist $(m_1,n_1) \in \mathbb{N}\times\mathbb{Z}$ such that $(m_0,n_0) \neq (m_1,n_1)$ and $c_e(m_0,n_0) = c_e(m_0,n_0,m_1,n_1)$.*

Proof. Let us fix $(m_0,n_0) \in \mathbb{N}\times\mathbb{Z}$, and set

$$a_0 := \min A_e|_{Z^{n_0}(M_e^{m_0})} \leq c_e(m_0,n_0,m_0+1,n_0) =: a_1.$$

Notice that $c_e(m_0,n_0) \in [a_0,a_1]$. By Lemma 3.5, there exists a finite union of critical circles $\mathcal{E} \subset \mathrm{crit}(A_e)$ such that, whenever $c_e(m_0,n_0,m_1,n_1) \in [a_0,a_1]$, $\mathcal{E}$ contains an essential family for $\mathcal{P}_e(m_0,n_0,m_1,n_1)$. We introduce the finite set of critical values

$$F := \big\{ A_e([w]) \ \big|\ [w] \in \mathcal{E} \big\}.$$

The value $c_e(m_0,n_0)$ is the infimum of those $c_e(m_0,n_0,m_1,n_1)$ belonging to the finite set F, and therefore it is a minimum. ∎

3.4. **The main multiplicity result.** Theorem 1.1 is an immediate consequence of the following more precise statement.

Theorem 3.7. *The set $I' \cap I_{\mathrm{finite}}$ is empty. Namely, for all energy values $e \in I'$, there are infinitely many periodic orbits with energy e.*

In the proof of Theorem 3.7, we will need the following abstract lemma established in [AMMP14, Lemma 2.5] for the free-period action functional. Being a local statement, such a lemma holds for the functional A_e as well (see Remark 2.1).

Lemma 3.8. *Every isolated critical circle $\mathcal{C} \subset \mathrm{crit}(A_e) \cap A_e^{-1}(c)$ has an arbitrarily small open neighborhood $\mathcal{U}$ such that the intersection $\mathcal{U} \cap \{A_e < c\}$ has only finitely many connected components.* ∎

Proof of Theorem 3.7. We assume by contradiction that there exists $e \in I' \cap I_{\mathrm{finite}}$. We set

$$a_0 := 0 < \left| \int_{S^2} \sigma \right| =: a_1.$$

Lemma 3.5 provides a finite union of critical circles

$$\mathcal{E} = \mathcal{C}_1 \cup ... \cup \mathcal{C}_s \subset \mathrm{crit}(A_e)$$

such that, whenever $c_e(m_0, n_0, m_1, n_1) \in [a_0, a_1]$, $\mathcal{E}$ contains an essential family for $\mathcal{P}_e(m_0, n_0, m_1, n_1)$. By Equation (3.15) and Lemma 3.6, for each $m \in \mathbb{N}$ there exist $n_m \in \mathbb{Z}$ and $(m'_m, n'_m) \in \mathbb{N} \times \mathbb{Z}$ such that

$$a_0 \le c_e(m, n_m) = c_e(m, n_m, m'_m, n'_m) < a_1.$$

In particular, $\mathcal{E}$ contains an essential family for $\mathcal{P}_e(m, n_m, m'_m, n'_m)$. For each $i = 1, ..., s$, we consider an open neighborhood $\mathcal{U}_i$ of the critical circle $\mathcal{C}_i$ given by Lemma 3.8. We define

$$\mathcal{F} := \bigcup_{i=1,\dots,s} \Big\{ \mathcal{V} \ \Big| \ \mathcal{V} \text{ is a connected component of } \mathcal{U}_i \cap \{A_e < A_e(\mathcal{C}_i)\} \Big\}.$$

Notice that $\mathcal{F}$ has finite cardinality according to Lemma 3.8. For each $m \in \mathbb{N}$, there exists $\mathcal{V}_m \in \mathcal{F}$ with the following property: there exists a path $\Theta_m \in \mathcal{P}_e(m, n_m, m'_m, n'_m)$ and $s_m \in [0, 1]$ such that the restriction $\Theta_m|_{[0,s_m]}$ is contained in the sublevel set $\{A_e < c_e(m, n_m, m'_m, n'_m)\}$, and $\Theta_m(s_m) \in \mathcal{V}_m$. Since $\mathcal{F}$ is finite, by the pigeonhole principle there exist distinct $m_1, m_2 \in \mathbb{N}$ such that $\mathcal{V}_{m_1} = \mathcal{V}_{m_2}$. In particular, $c_e(m_1, n_{m_1}, m'_{m_1}, n'_{m_1}) = c_e(m_2, n_{m_2}, m'_{m_2}, n'_{m_2})$.

Consider the path $\Theta : [0, 1] \to \widetilde{\mathcal{M}}$ obtained by concatenation of three paths: the restricted path $\Theta_{m_1}|_{[0,s_{m_1}]}$, some path connecting $\Theta_{m_1}(s_{m_1})$ with $\Theta_{m_2}(s_{m_2})$ within $\mathcal{V}_{m_1}$, and the restricted path $\Theta_{m_2}|_{[0,s_{m_2}]}$ traversed in the opposite direction. By construction, $\Theta \in \mathcal{P}_e(m_1, n_{m_1}, m_2, n_{m_2})$. However,

$$\max A_e \circ \Theta < c_e(m_1, n_{m_1}, m'_{m_1}, n'_{m_1}) = c_e(m_1, n_{m_1}) \le c_e(m_1, n_{m_1}, m_2, n_{m_2}),$$

which contradicts the definition of $c_e(m_1, n_{m_1}, m_2, n_{m_2})$. ∎

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