# Bifurcation of gravity-capillary Stokes waves with constant vorticity

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**Abstract.** We consider the gravity-capillary water waves equations of a 2D fluid with constant vorticity. Using variational methods we prove the bifurcation of steady periodic traveling water waves for *all* the values of gravity, surface tension, constant vorticity, depth and wavelenght, extending previous results valid for restricted values of the parameters. We parametrize the bifurcating Stokes waves either with their speed or their momentum.

*Key words*: Bifurcation of Stokes waves, variational methods, gravity-capillary water waves, critical point theory.
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## 1 Introduction and main results

A very classical fluid mechanics problem regards the search for traveling surface waves. In this paper we consider the Euler equations for a 2-dimensional incompressible and inviscid fluid with constant vorticity $\gamma$, under the action of gravity $g > 0$ and surface tension $\kappa \geq 0$ at the free surface. The fluid occupies the region

$$\mathcal{D}_{\eta,\mathtt{h}} := \left\{(x,y) \in (\lambda\mathbb{T}) \times \mathbb{R} \ : \ -\mathtt{h} < y < \eta(t,x)\right\}, \quad \mathbb{T} := \mathbb{T}_x := \mathbb{R}/(2\pi\mathbb{Z}), \tag{1.1}$$

with a, possibly infinite, depth $\mathtt{h} > 0$ and space periodic boundary conditions with wavelength $2\pi\lambda > 0$.

The goal is to show that variational methods, based on the Hamiltonian formulation of the water waves equations [49, 14, 9, 45], allow to prove the bifurcation of steady periodic traveling water waves -called Stokes waves- for *all* the values of gravity $g > 0$, surface tension $\kappa \geq 0$, constant vorticity $\gamma \in \mathbb{R}$, depth $\mathtt{h} \in (0, +\infty]$ and wavelenght $2\pi\lambda > 0$ (clearly not all the physical parameters $g, \mathtt{h}, \kappa, \gamma, \lambda$ are independent). Previous results as [43, 30], based on the use of the Crandall-Rabinowitz bifurcation theorem from the simple eigenvalue, restrict the range of allowed parameters. We shall prove the existence of non-trivial Stokes waves parametrized by the speed, see Theorem 1.2, or the momentum, see Theorem 1.3.

The literature concerning steady traveling wave solutions is enormous. We refer to [8, 21] for extended presentations. Here we only mention that, after the pioneering work of Stokes [39], the first rigorous construction of small amplitude space periodic steady traveling waves is due to Nekrasov [33], Levi-Civita [27] and Struik [40] for irrotational 2D flows under the action of gravity. Later Zeidler [50] considered the effect of capillarity, see also Jones-Toland [24], and Dubreil-Jacotin [15], Goyon [18] of vorticity. More recent results for capillary-gravity waves with vorticity have been given in Wahlén [44, 43] and Martin [30]. All these works deal with 2D water waves, and can ultimately be deduced by the Crandall-Rabinowitz bifurcation theorem from a simple eigenvalue.

These local bifurcation results have been extended to global branches of steady traveling waves, as started in the celebrated works of Keady-Norbury [25], Toland [42], Amick-Fraenkel-Toland [2], McLeod [31], Plotinkov [35] for irrotational flows and Constantin-Strauss

[10], Constantin-Strauss-Varvaruca [11], for fluids with vorticity. See also the recent works by Wahlén-Weber [46] and Kozlov-Lokharu [26] for general vorticity.

For three dimensional irrotational fluids, bifurcation of small amplitude bi-periodic traveling waves has been proved in Reeder-Shinbrot [38], Craig-Nicholls [12, 13] for gravity-capillary waves by variational methods and by Iooss-Plotnikov [22, 23] for gravity waves (this is a small divisor problem). We also quote the results [28, 20] for doubly periodic gravity-capillary Beltrami flows.

We finally mention that in the last years also the existence of quasi-periodic traveling Stokes waves –which are the nonlinear superposition of Stokes waves moving with rationally independent speeds– has been proved in [4, 17, 5] by means of KAM methods.

The results of the present paper, Theorems 1.2, 1.3, are not covered by Craig-Nicholls [12] as we add the constant vorticity for $2D$ fluids and we parametrize the Stokes waves also with their speed. This requires a different critical point theory. With respect to Martin [30] we cover all the possible bifurcation speeds, also the resonant ones, that we characterize in Proposition 1.1. Let us now present rigorously the results and the techniques.

**The water waves equations.** In the sequel, with no loss of generality, we set $\lambda = 1$. The unknowns of the problem are the free surface $y = \eta(t, x)$ of the time dependent domain $\mathcal{D}_{\eta,\mathtt{h}}$ in (1.1) and the divergence free velocity field $\begin{pmatrix} u(t,x,y) \\ v(t,x,y) \end{pmatrix}$. If the fluid has constant vorticity $v_x - u_y = \gamma$, the velocity field is the sum of the Couette flow $\begin{pmatrix} -\gamma y \\ 0 \end{pmatrix}$, which carries all the vorticity $\gamma$ of the fluid, and an irrotational field $\nabla_{x,y}\Phi(t, x, y)$. Given $\psi(t, x) := \Phi(t, x, \eta(t, x))$ one recovers $\Phi$ by solving the elliptic problem

$$\Delta\Phi = 0 \ \text{ in } \mathcal{D}_{\eta,\mathtt{h}} \,, \quad \Phi = \psi \ \text{ at } y = \eta(t,x) \,, \quad \Phi_y \to 0 \ \text{ as } y \to -\mathtt{h} \,.$$

Imposing that the fluid particles at the free surface remain on it along the evolution (kinematic boundary condition), and that the pressure of the fluid is constant at the free surface (dynamic boundary condition), the time evolution of the fluid is determined by the system of equations

$$\begin{cases} \eta_t = G(\eta)\psi + \gamma\eta\eta_x \\ \psi_t = -g\eta - \dfrac{\psi_x^2}{2} + \dfrac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} + \kappa\partial_x\Big(\dfrac{\eta_x}{\sqrt{1+\eta_x^2}}\Big) + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi \,, \end{cases} \tag{1.2}$$

where $G(\eta)$ is the Dirichlet-Neumann operator

$$G(\eta)\psi := G(\eta,\mathtt{h})\psi := (-\Phi_x\eta_x + \Phi_y)|_{y=\eta(x)} \,.$$

In (1.2) $(\partial_x^{-1}f)(x)$ denotes the unique primitive with zero average of a $2\pi$-periodic zero average function $f(x)$. It turns out that $G(\eta)\psi$ has zero average. As consequence the average of $\eta(x)$, $\langle\eta\rangle := \eta_0 := \frac{1}{2\pi}\int_{\mathbb{T}}\eta(x)\,dx$ is a prime integral of (1.2). Note also that, since $G(\eta)[1] = 0$ vanishes on the constants, the vector field in the right hand side of (1.2) does not depend on $\psi_0 = \langle\psi\rangle$.

**Hamiltonian structure.** As observed in the irrotational case by Zakharov [49], Craig-Sulem [14], and in presence of constant vorticity by Constantin-Ivanov-Prodanov [9] and Wahlén [45], the water waves equations (1.2) are the Hamiltonian system with a non-canonical structure

$$\eta_t = \nabla_\psi H(\eta,\psi) \,, \quad \psi_t = (-\nabla_\eta + \gamma\partial_x^{-1}\nabla_\psi)H(\eta,\psi) \,, \tag{1.3}$$

where $\nabla$ denotes the $L^2$-gradient, with Hamiltonian

$$H(\eta,\psi) = \int_{\mathbb{T}} \frac{1}{2}\Big(\psi\, G(\eta)\psi + g\eta^2\Big) + \kappa\big(\sqrt{1+\eta_x^2} - 1\big) + \frac{\gamma}{2}\big(-\psi_x\eta^2 + \frac{\gamma}{3}\eta^3\big)dx \,. \tag{1.4}$$

The equations (1.2) simplify considerably introducing as in Wahlén [43] the variable

$$\zeta := \psi - \frac{\gamma}{2}\partial_x^{-1}\big(\eta - \langle\eta\rangle\big) = \psi - \frac{\gamma}{2}\partial_x^{-1}\Pi_0^\perp \eta \tag{1.5}$$

where $\Pi_0^\perp$ is the projector on the zero average functions, $\Pi_0^\perp := \mathrm{Id} - \Pi_0$ and $\Pi_0\eta := \langle\eta\rangle$. Actually, under the linear change of variable

$$\begin{pmatrix}\eta\\ \psi\end{pmatrix} = W\begin{pmatrix}\eta\\ \zeta\end{pmatrix}, \quad W = \begin{pmatrix} I & 0\\ \frac{\gamma}{2}\partial_x^{-1}\Pi_0^\perp & I\end{pmatrix}, \quad W^{-1} = \begin{pmatrix} I & 0\\ -\frac{\gamma}{2}\partial_x^{-1}\Pi_0^\perp & I\end{pmatrix}, \tag{1.6}$$

the Hamiltonian system (1.3) assumes the canonical Darboux form

$$\partial_t\begin{pmatrix}\eta\\ \zeta\end{pmatrix} = J\begin{pmatrix}\nabla_\eta\mathcal{H}(\eta,\zeta)\\ \nabla_\zeta\mathcal{H}(\eta,\zeta)\end{pmatrix} \qquad \text{where} \qquad J := \begin{pmatrix}0 & I\\ -I & 0\end{pmatrix} \tag{1.7}$$

is the canonical Poisson tensor and the new Hamiltonian is $\mathcal{H}(\eta,\zeta) := H\circ W(\eta,\zeta)$ .

The symplectic form associated to the Hamiltonian structure of (1.7) is

$$\boldsymbol{\mathcal{W}}\left(\begin{pmatrix}\eta\\ \zeta\end{pmatrix}, \begin{pmatrix}\eta_1\\ \zeta_1\end{pmatrix}\right) = \left\langle J^{-1}\begin{pmatrix}\eta\\ \zeta\end{pmatrix}, \begin{pmatrix}\eta_1\\ \zeta_1\end{pmatrix}\right\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} \eta(x)\zeta_1(x) - \eta_1(x)\zeta(x)\,dx \tag{1.8}$$

where $\langle f,g\rangle := \frac{1}{2\pi}\int_{\mathbb{T}} f(x)g(x)dx$ denotes the $L^2(\mathbb{T},\mathbb{R}^2)$ real scalar product, so that

$$\mathrm{d}\mathcal{H}(u)[\cdot] = \boldsymbol{\mathcal{W}}(J\nabla\mathcal{H}(u),\cdot)\,, \tag{1.9}$$

i.e. the Hamiltonian vector $J\nabla\mathcal{H}(u)$ field is the symplectic gradient of $\mathcal{H}(u)$. Note that the vector field $J\nabla\mathcal{H}(\eta,\zeta)$ in (1.7) does not depend on $\zeta_0 = \langle\zeta\rangle$ and the first component has zero average, so that the average $\langle\eta\rangle$ is a prime integral of (1.7), as well as for (1.2).

**Reversible structure and $O(2)$-symmetry.** The Hamiltonian system (1.7) possesses both a $\mathbb{Z}_2$ and an $\mathbb{S}^1$-symmetry. Indeed it is reversible, namely

$$\mathcal{H}\circ\mathscr{S} = \mathcal{H} \qquad \text{where} \qquad \mathscr{S}\begin{pmatrix}\eta\\ \zeta\end{pmatrix}(x) := \begin{pmatrix}\eta(-x)\\ -\zeta(-x)\end{pmatrix}, \tag{1.10}$$

(note that $\mathscr{S} = \mathscr{S}^{-1}$), and, since the bottom of the fluid domain is flat, it is space invariant, namely

$$\mathcal{H}\circ\tau_\theta = \mathcal{H} \qquad \text{where} \qquad \tau_\theta\begin{pmatrix}\eta\\ \zeta\end{pmatrix}(x) := \begin{pmatrix}\eta(x-\theta)\\ \zeta(x-\theta)\end{pmatrix}, \quad \forall\theta\in\mathbb{R}\,. \tag{1.11}$$

Note that $(\tau_\theta)_{\theta\in\mathbb{R}}$ is a one parameter group of symplectic transformations. According to Noether theorem the associated prime integral of (1.7) is the momentum

$$\mathcal{I}(\eta,\zeta) := \int_{\mathbb{T}} \eta_x(x)\zeta(x)\,dx\,, \tag{1.12}$$

since the Hamiltonian vector field generated by $\mathcal{I}$ is

$$J\nabla\mathcal{I}(\eta,\zeta) = \partial_x\begin{pmatrix}\eta\\ \zeta\end{pmatrix}, \tag{1.13}$$

which is the generator of the group of the translations (1.11). Also the momentum $\mathcal{I}$ clearly satisfies

$$\mathcal{I}\circ\mathscr{S} = \mathcal{I}\,, \quad \mathcal{I}\circ\tau_\theta = \mathcal{I}\,,\ \forall\theta\in\mathbb{R}\,. \tag{1.14}$$

These joint symmetries actually amount to the fact that $\mathcal{H}$ and $\mathcal{I}$ are invariant under the action of the orthogonal group $O(2) \cong \mathbb{S}^1 \rtimes \mathbb{Z}_2$, cfr. Remark 2.1.

**Traveling waves.** We seek for steady traveling waves of (1.7), i.e. solutions of the form $\eta(x-ct)$ and $\zeta(x-ct)$ where $\eta(x), \zeta(x)$ are $2\pi$-periodic. Substituting inside (1.7) we obtain

$$-c\,\partial_x \begin{pmatrix}\eta\\ \zeta\end{pmatrix} = J\begin{pmatrix}\nabla_\eta \mathcal{H}(\eta,\zeta)\\ \nabla_\zeta \mathcal{H}(\eta,\zeta)\end{pmatrix},$$

which, in view of (1.2), (1.6), (1.13) is equivalent to find solutions $c \in \mathbb{R}$ and $u = (\eta, \zeta)$ of the nonlinear equation

$$\mathcal{F}(c,u) := c\partial_x u + J\nabla\mathcal{H}(u) = J(\nabla\mathcal{H} + c\nabla\mathcal{I})(u) = 0\,. \tag{1.15}$$

We regard $\mathcal{F}(c,\cdot)$ as a map defined in a dense subset $X \subset L^2(\mathbb{T}) \times L^2_0(\mathbb{T})$, that we define below in (1.18),

$$\mathcal{F}\colon \mathbb{R}\times X \to Y \subset L^2_0(\mathbb{T}) \times L^2(\mathbb{T})\,, \quad (c,u) \mapsto \mathcal{F}(c,u)\,, \tag{1.16}$$

where $L^2_0(\mathbb{T})$ is the subspace of $L^2(\mathbb{T})$ of zero average functions. Since $J$ is invertible, the equation (1.15) is equivalent to search critical points, i.e. equilibria, of the Hamiltonian

$$\Psi(c,\cdot) : X \to \mathbb{R}\,, \quad \Psi(c,u) := (\mathcal{H} + c\mathcal{I})(u)\,, \tag{1.17}$$

for some value of the moving frame speed $c$.

By the group symmetries (1.11), (1.10), (1.14), if $u$ is a Stokes wave solution of (1.15) then each translated function $\tau_\theta u$ and reflected one $\mathscr{S}u$ are solutions as well. We shall say that two non-trivial Stokes waves solutions of (1.15) are *geometrically distinct* if they are not obtained by applying the translation operator $\tau_\theta$ or the reflection operator $\mathscr{S}$ to the other one.

**Functional setting.** We choose, for $\sigma > 0$, $s \geq 7/2$, $s + \frac{1}{2} \in \mathbb{N}$,

$$X := H^{\sigma,s} \times H^{\sigma,s}_0 := H^{\sigma,s}(\mathbb{T}) \times H^{\sigma,s}_0(\mathbb{T})\,, \tag{1.18}$$

where $H^{\sigma,s}(\mathbb{T})$ is the space of $2\pi$-periodic analytic functions $u(x) = \sum_{k\in\mathbb{Z}} u_k e^{\mathrm{i}kx}$ with norm

$$\|u\|^2_{\sigma,s} := \sum_{k\in\mathbb{Z}} |u_k|^2 \langle k\rangle^{2s} e^{2\sigma|k|} < +\infty\,, \quad \langle k\rangle := \max\{1,|k|\}\,,$$

and $H^{\sigma,s}_0 := H^{\sigma,s} \cap L^2_0(\mathbb{T})$. For any $\sigma \geq 0$ and $s > 1/2$ each space $H^{\sigma,s}(\mathbb{T})$ is an algebra with respect to the product of functions. The target space in (1.16) is, in view of (1.2) and (1.5),

$$Y := H^{\sigma,s-1}_0(\mathbb{T}) \times H^{\sigma,s-2}(\mathbb{T}) \text{ if } \kappa > 0\,, \quad Y := H^{\sigma,s-1}_0(\mathbb{T}) \times H^{\sigma,s-1}(\mathbb{T}) \text{ if } \kappa = 0\,. \tag{1.19}$$

Note that $\mathcal{F}(c,0) = 0$ for any $c \in \mathbb{R}$. We are going to prove, by means of variational arguments, that, for *any* value of the parameters

$$(g, \mathtt{h}, \kappa, \gamma) \in (0,+\infty) \times (0,+\infty) \times [0,+\infty) \times \mathbb{R}\,,$$

*any* point $(c_*, 0)$ where $\mathcal{L}_{c_*} := \mathrm{d}_u\mathcal{F}(c_*,0)$ is not invertible (this is a necessary condition for bifurcation) is actually a point of bifurcation of non-trivial Stokes waves solutions of $\mathcal{F}(c,u) = 0$. We now present in detail the main results and techniques of proof.

**Main results.** In Section 2 we diagonalize the operator $\mathrm{d}_u\mathcal{F}(c,0)$, written explicitly in (2.1), and we prove that all the possible speeds of bifurcation form the set

$$\mathsf{C} := \left\{ \frac{\Omega_j(g,\mathtt{h},\kappa,\gamma)}{j} : j \in \mathbb{Z}\setminus\{0\} \right\} \tag{1.20}$$

where $\Omega_\xi := \Omega_\xi(g, \mathtt{h}, \kappa, \gamma)$ is the dispersion relation of the gravity-capillary water waves equations with constant vorticity

$$\Omega_\xi := \begin{cases} \frac{\gamma}{2}\tanh(\mathtt{h}\xi) + \sqrt{(g+\kappa\xi^2)\xi\tanh(\mathtt{h}\xi) + \frac{\gamma^2}{4}\tanh^2(\mathtt{h}\xi)}\,, & \text{if } \mathtt{h} < +\infty \\ \frac{\gamma}{2}\mathrm{sign}(\xi) + \sqrt{(g+\kappa\xi^2)|\xi| + \frac{\gamma^2}{4}}\,, & \text{if } \mathtt{h} = +\infty\,. \end{cases} \tag{1.21}$$

Sometimes, in the Stokes waves literature, the function $j \mapsto \frac{\Omega_j}{j}$ is called the dispersion relation as well.

Then we fix any integer $j_* \in \mathbb{Z} \setminus \{0\}$ and we consider the bifurcation speed

$$c_* = \frac{\Omega_{j_*}}{j_*} \in \mathsf{C}\,, \tag{1.22}$$

which is associated to a 2-dimensional real eigenspace of $\mathcal{L}_{c_*}$ with zero eigenvalue, see (2.16). In order to see if the Kernel of $\mathcal{L}_{c_*}$ is 2 dimensional or higher dimensional, we have to determine if there exist other integers $j$ such that

$$\frac{\Omega_j}{j} = c_* = \frac{\Omega_{j_*}}{j_*}\,. \tag{1.23}$$

In Proposition 1.1 we fully characterize the values of $g, \mathtt{h}, \kappa, \gamma$ such that dim Ker $\mathcal{L}_{c_*} = 2$ (non-resonant case) or higher dimensional, in such a case it turns out that dim Ker $\mathcal{L}_{c_*} = 4$ (resonant case). We introduce, for any $\mathtt{h} < +\infty$, the "vorticity modified-Bond" numbers

$$\mathtt{B}_\pm(g, \mathtt{h}, \kappa, \gamma) := \frac{\kappa}{g\mathtt{h}^2} - \frac{\mathtt{h}\gamma^2}{6g}\Big(1 \pm \sqrt{1 + \frac{4g}{\mathtt{h}\gamma^2}}\,\Big)\,, \tag{1.24}$$

which reduce, in the irrotational case $\gamma = 0$, to the "classical" Bond number $\mathtt{B}_\pm(g, \mathtt{h}, \kappa, 0) = \frac{\kappa}{g\mathtt{h}^2}$. In Section 3 we prove the following result.

**Proposition 1.1. (Kernel of $\mathcal{L}_{c_*}$).** *For any $g > 0$, $\kappa \geq 0$, $\mathtt{h} \in (0, +\infty]$, $\gamma \in \mathbb{R}$, $j_* \in \mathbb{Z} \setminus \{0\}$, let $c_* := c_*(g, \mathtt{h}, \kappa, \gamma, j_*)$ be the speed defined in* (1.22)*. The following holds true: if*

*(i) either $\mathtt{h} < \infty$, $\kappa > 0$ and the vorticity-Bond numbers $\mathtt{B}_\pm(g, \mathtt{h}, \kappa, \gamma)$ in* (1.24) *fulfill*

$$\mathtt{B}_+(g, \mathtt{h}, \kappa, \gamma) < \tfrac{1}{3} \ \textit{if}\ c_* > 0\,, \quad \mathtt{B}_-(g, \mathtt{h}, \kappa, \gamma) < \tfrac{1}{3} \ \textit{if}\ c_* < 0\,;$$

*(ii) or $\mathtt{h} = +\infty$ and $\kappa > 0$;*

*then the kernel of the operator $\mathcal{L}_{c_*}$ has dimension either two or four. In all other cases* $\dim\ker\mathcal{L}_{c_*} = 2$*. For any pair of integers $1 \leq |j_*| < |j|$, for any $g > 0$, $\mathtt{h} \in (0, +\infty]$, $\gamma \in \mathbb{R}$, there exists $\kappa > 0$ such that* (1.23) *holds, in particular* $\dim\ker\mathcal{L}_{c_*} = 4$.

The non-resonant cases when $\dim\ker\mathcal{L}_{c_*} = 2$ have been studied in [43, 30] by means of the Crandall-Rabinowitz bifurcation theorem, as we report in paragraph 4. In this case a minor improvement is to prove also the analyticity of the Stokes waves. The main novelty is to prove the bifurcation of Stokes waves in the resonant case when the Kernel of $\mathrm{d}_u\mathcal{F}(c_*, 0)$ is 4 dimensional, by means In Section 5 we use critical point theory.

Our first result proves the existence of solutions of $\mathcal{F}(c, u) = 0$ for any fixed speed $c$ close to $c_*$. According to all the possible values of $g, \mathtt{h}, \kappa, \gamma$ and $j_*$, the non-trivial Stokes wave solutions of (1.2) have speed either $c = c_*$ (case $(i)$), or $c < c_*$ (sub-critical bifurcation) or $c > c_*$ (super-critical bifurcation) as described in cases $(ii)$-$(iii)$ below.

**Theorem 1.2. (Stokes waves with fixed speed)** *For any* $(g, \mathtt{h}, \kappa, \gamma) \in (0, +\infty) \times (0, +\infty] \times [0, +\infty) \times \mathbb{R}$ *and any integer* $j_* \in \mathbb{Z} \setminus \{0\}$*, the following alternatives may occur. Either*

*(i)* $u = 0$ *is a non-isolated solution of* $\mathcal{F}(c_*, u) = 0$*;*

*or*

*(ii) there is a one sided neighborhood* $\mathcal{U}$ *of* $c_*$ *such that for any* $c \in \mathcal{U} \setminus \{c_*\}$ *the equation* $\mathcal{F}(c, u) = 0$ *possesses at least two geometrically distinct non-trivial solutions, which tend to* $0$ *as* $c \to c_*$*;*

*or*

*(iii) there is a neighborhood* $\mathcal{U}$ *of* $c_*$ *such that for any* $c \in \mathcal{U} \setminus \{c_*\}$ *the equation* $\mathcal{F}(c, u) = 0$ *possesses at least one non-trivial solution, which tends to* $0$ *as* $c \to c_*$*.*

The proof, given in Section 5.1, is based on searching for mountain pass critical points of the reduced Hamiltonian obtained after a variational Lyapunov-Schmidt reduction à la Fadell-Rabinowitz [16, 36]. The most delicate case is *(iii)* for which we provide complete and self-contained proofs in Appendix A. We hope that these powerful techniques could be more effectively used for bifurcation problems for fluid PDEs.

In Section 5.2 we look for solutions parametrized by the momentum.

**Theorem 1.3. (Stokes waves with fixed momentum)** *For any* $(g, \mathtt{h}, \kappa, \gamma) \in (0, +\infty) \times (0, +\infty] \times [0, +\infty) \times \mathbb{R}$ *and any integer* $j_* \in \mathbb{Z} \setminus \{0\}$*, for any* $a \in (-a_0, a_0)$ *small enough, with* $\mathrm{sign}(a) = -\mathrm{sign}(j_*)$*, there exist at least two geometrically distinct non-trivial solutions*

$$(c_i(a), u_i(a)), \quad i = 1, 2, \quad \textit{of the equation} \quad \mathcal{F}(c, u) = 0,$$

*with momentum* $\mathcal{I}(u_i(a)) = a$ *and* $(c_i(a), u_i(a)) \to (c_*, 0)$ *as* $a \to 0$*.*

In this case the proof is reduced to search for critical points of a reduced Hamiltonian on a sphere-like manifold, in the spirit of Weinstein-Moser resonant center theorems [32, 47, 48], see also [3], and Craig-Nicholls [12].

The paper is organized as follows. In Section 2 we characterize all the bifurcation speeds where $\mathrm{d}_u\mathcal{F}(c, 0)$ is not invertible, i.e the set (1.20). In Section 3 we prove Proposition 1.1 which establishes, according to the values of $g, \kappa, \gamma, \mathtt{h}$, if the Kernel of $\mathrm{d}_u\mathcal{F}(c_*, 0)$ is 2 dimensional (non-resonant case) or 4 dimensional (resonant case). In Section 4 we perform the variational Lyapunov-Schmidt reduction. In section 5 we prove the bifurcation of Stokes waves in the resonant case either with fixed speed (Theorem 1.2) or with fixed momentum (Theorem 1.3). In Appendix A we prove the existence of Palais-Smale sequences at the mountain pass level.

## 2 The linearized operator

We study the linearized operator

$$\mathrm{d}_u\mathcal{F}(c, 0) := \mathcal{L}_c := c\partial_x + J\mathrm{d}\nabla\mathcal{H}(0) \tag{2.1}$$

where, in view of (1.4), (1.6),

$$\mathrm{d}\nabla\mathcal{H}(0) := W^\top \mathrm{d}\nabla H(0) W = \begin{pmatrix} g - \kappa\partial_x^2 - \frac{\gamma^2}{4}\Pi_0^\perp\partial_x^{-1}G(0)\partial_x^{-1}\Pi_0^\perp & -\frac{\gamma}{2}\Pi_0^\perp\partial_x^{-1}G(0) \\ \frac{\gamma}{2}G(0)\partial_x^{-1}\Pi_0^\perp & G(0) \end{pmatrix}$$

and

$$G(0) = D \tanh(\mathtt{h}D) \ \text{ if } \mathtt{h} < +\infty\,, \qquad G(0) = |D| \ \text{ if } \mathtt{h} = +\infty\,,$$

is the Dirichlet-Neumann operator at the flat surface. Note that the real operator $\mathrm{d}\nabla\mathcal{H}(0)$ is symmetric and that

$$c\partial_x = J\mathrm{d}\nabla\mathcal{I}(0)\,, \quad \mathrm{d}\nabla\mathcal{I}(0) = \begin{pmatrix} 0 & -\partial_x \\ \partial_x & 0 \end{pmatrix}\,, \tag{2.2}$$

so the linear operator $\mathcal{L}_c$ is Hamiltonian, i.e. of the form $JA$ where $A$ is a symmetric operator.

In order to diagonalize $\mathcal{L}_c$, we conjugate it with the symplectic map

$$\mathcal{M} := \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix} \quad , \quad M := \left( \frac{G(0) + g\Pi_0}{g - \kappa\partial_x^2 - \frac{\gamma^2}{4}\Pi_0^\perp \partial_x^{-1} G(0)\partial_x^{-1}\Pi_0^\perp} \right)^{\frac{1}{4}}\,, \tag{2.3}$$

i.e. $\mathcal{M}^\top J \mathcal{M} = J$, obtaining the Hamiltonian operator

$$\mathcal{L}_c^{(1)} := \mathcal{M}^{-1}\mathcal{L}_c\mathcal{M} = c\partial_x + J \begin{pmatrix} \omega(D) + g\Pi_0 & -\frac{\gamma}{2}\Pi_0^\perp\partial_x^{-1}G(0) \\ \frac{\gamma}{2}G(0)\partial_x^{-1}\Pi_0^\perp & \omega(D) \end{pmatrix}$$

where $\omega(D)$ is the Fourier multiplier with symbol

$$\omega(\xi) := \begin{cases} \sqrt{(g + \kappa\xi^2)\xi\tanh(\mathtt{h}\xi) + \frac{\gamma^2}{4}\tanh^2(\mathtt{h}\xi)} & \text{if } \mathtt{h} < +\infty \\ \sqrt{(g + \kappa\xi^2)|\xi| + \frac{\gamma^2}{4}} & \text{if } \mathtt{h} = +\infty\,. \end{cases}$$

Note that $\omega(0) = 0$. Next we pass to complex coordinates via the invertible transformation

$$\mathcal{C} := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}\,, \quad \mathcal{C}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}\,, \tag{2.4}$$

obtaining the complex Hamiltonian operator

$$\mathcal{L}_c^{(2)} := \mathcal{C}^{-1}\mathcal{L}_c^{(1)}\mathcal{C} = c\partial_x + J_c\left(\boldsymbol{\Omega} + g\boldsymbol{\Pi}_0\right) \quad \text{where} \quad J_c := \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \tag{2.5}$$

is the complex Poisson tensor, and $\boldsymbol{\Omega}$, $\boldsymbol{\Pi}_0$ are the selfadjoint operators

$$\boldsymbol{\Omega} := \begin{pmatrix} \Omega(D) & 0 \\ 0 & \overline{\Omega(D)} \end{pmatrix}\,, \qquad \boldsymbol{\Pi}_0 := \frac{1}{2}\Pi_0\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\,, \tag{2.6}$$

with Fourier multiplier

$$\Omega(D) := \omega(D) + \mathrm{i}\frac{\gamma}{2}G(0)\partial_x^{-1}$$

and real symbol $\Omega_\xi := \Omega_\xi(g, \mathtt{h}, \kappa, \gamma)$ defined in (1.21). In (2.6) the operator $\overline{\Omega(D)}$ is the Fourier multiplier with symbol $\Omega_{-\xi}$.

Note that if the surface tension $\kappa > 0$ then the dispersion relation $\xi \mapsto \Omega_\xi$ in (1.21) is a symbol of order $3/2$ with asymptotic expansion

$$\Omega_\xi = \sqrt{\kappa}|\xi|^{3/2} + a_0(\xi) \qquad \text{where} \qquad a_0(\xi) \in S^0 \tag{2.7}$$

is a Fourier multiplier of order 0. If $\kappa = 0$ then $\Omega_\xi$ in (1.21) is a symbol of order $1/2$.

The operator $\mathcal{L}_c^{(2)}$ in (2.5) acts on the zero average functions as

$$\begin{pmatrix} c\partial_x - \mathrm{i}\Omega(D) & 0 \\ 0 & c\partial_x + \mathrm{i}\overline{\Omega(D)} \end{pmatrix} \tag{2.8}$$

and, in the Fourier basis $\{e^{\mathrm{i}jx}\}_{j\in\mathbb{Z}\setminus\{0\}}$, as the diagonal operator $\mathcal{L}_c^{(2)} = \mathrm{diag}(\mathcal{L}_j)_{j\in\mathbb{Z}\setminus\{0\}}$ where

$$\mathcal{L}_j := \begin{pmatrix} \mathrm{i}(cj-\Omega_j) & 0 \\ 0 & \mathrm{i}(cj+\Omega_{-j}) \end{pmatrix}, \ \forall j\in\mathbb{Z}\setminus\{0\}. \tag{2.9}$$

Recalling (2.1), the possible speeds of bifurcation $c\in\mathbb{R}$ are those for which $\mathcal{L}_c^{(2)}$ has a nontrivial kernel, namely, in view of (2.9), the set $\mathsf{C}$ defined in (1.20).

**Decomposition of the phase space in symplectic subspaces invariant under $\mathcal{L}_c$.** We decompose any function $(\eta,\zeta)$ of the space $L^2(\mathbb{T})\times L^2(\mathbb{T})$ as

$$u(x) = \begin{pmatrix} \eta(x) \\ \zeta(x) \end{pmatrix} = \eta_0 v_0^{(1)} + \zeta_0 v_0^{(2)} + \sum_{j\in\mathbb{Z}\setminus\{0\}} \alpha_j v_j^{(1)}(x) + \beta_j v_j^{(2)}(x) \tag{2.10}$$

where, denoting $M_j$ the real symbol of the Fourier multiplier in (2.3),

$$v_0^{(1)} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ v_0^{(2)} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_j^{(1)}(x) := \begin{pmatrix} M_j\cos(jx) \\ M_j^{-1}\sin(jx) \end{pmatrix}, \ v_j^{(2)}(x) := \begin{pmatrix} -M_j\sin(jx) \\ M_j^{-1}\cos(jx) \end{pmatrix}, \tag{2.11}$$

for any $j\in\mathbb{Z}\setminus\{0\}$, and

$$\alpha_j := \alpha_j(u) = \boldsymbol{\mathcal{W}}(u, v_j^{(2)}), \quad \beta_j := \beta_j(u) = -\boldsymbol{\mathcal{W}}(u, v_j^{(1)}). \tag{2.12}$$

The 2-dimensional real vector spaces

$$V_0 := \langle v_0^{(1)}, v_0^{(2)}\rangle, \quad V_j := \left\langle v_j^{(1)}, v_j^{(2)} \right\rangle, \ j\in\mathbb{Z}\setminus\{0\}, \tag{2.13}$$

are invariant under the action of $\mathcal{L}_c$,

$$\mathcal{L}_c : V_j \to V_j, \quad \forall j\in\mathbb{Z}, \quad \forall c\in\mathbb{R}, \tag{2.14}$$

as

$$\mathcal{L}_c v_0^{(1)} = -g v_0^{(2)}, \tag{2.15}$$

and, for any $j\neq 0$,

$$\mathcal{L}_c v_j^{(1)} = (cj-\Omega_j)v_j^{(2)}, \ \mathcal{L}_c v_j^{(2)} = -(cj-\Omega_j)v_j^{(1)}, \quad \partial_x v_j^{(1)} = j v_j^{(2)}, \ \partial_x v_j^{(2)} = -j v_j^{(1)}. \tag{2.16}$$

The first identities in (2.16) directly follow by (2.5) and (2.8) since each subspace $V_j$ in (2.13) is, for any $j\neq 0$, the image under the map $\mathcal{MC}$ defined in (2.3), (2.4) of the 1-d complex vector subspace $\begin{pmatrix} ze^{\mathrm{i}jx} \\ \overline{z}e^{-\mathrm{i}jx} \end{pmatrix}$, $z\in\mathbb{C}$, namely

$$v_j^{(1)}(x) = \mathcal{MC}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{\mathrm{i}jx} \\ e^{-\mathrm{i}jx} \end{pmatrix}, \quad v_j^{(2)}(x) = \mathcal{MC}\frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{i}e^{\mathrm{i}jx} \\ -\mathrm{i}e^{-\mathrm{i}jx} \end{pmatrix}. \tag{2.17}$$

Each subspace $V_j$ is symplectic since the symplectic form $\boldsymbol{\mathcal{W}}$ in (1.8) restricted to $V_j$ reads

$$\boldsymbol{\mathcal{W}}_{|V_j} \equiv \mathrm{d}\alpha_j \wedge \mathrm{d}\beta_j \qquad \text{as} \qquad \mathcal{W}(v_j^{(1)}, v_j^{(2)}) = 1, \quad \forall j\in\mathbb{Z}. \tag{2.18}$$

Furthermore the subspaces $V_j$ are pairwise symplectic orthogonal, namely

$$V_k \perp^{\boldsymbol{\mathcal{W}}} V_j, \qquad \forall k\neq j,$$

but they are *not* all orthogonal, for example $V_j$ and $V_{-j}$.

The spaces $V_j$ are invariant with respect the involution $\mathscr{S}$ in (1.10) and the translations $\tau_\theta$ in (1.11), namely $\mathscr{S} V_j \subset V_j$, $\tau_\theta V_j \subset V_j$. Since $\mathscr{S} v_j^{(1)} = v_j^{(1)}$ and $\mathscr{S} v_j^{(2)} = -v_j^{(2)}$, and

$$\tau_\theta v_j^{(1)} = \cos(j\theta) v_j^{(1)} - \sin(j\theta) v_j^{(2)}\,, \quad \tau_\theta v_j^{(2)} = \sin(j\theta) v_j^{(1)} + \cos(j\theta) v_j^{(2)}\,,$$

in the coordinates $(\alpha_j, \beta_j)$, they read

$$\mathscr{S} : \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix}\,, \quad \tau_\theta : \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} \mapsto \underbrace{\begin{pmatrix} \cos(j\theta) & \sin(j\theta) \\ -\sin(j\theta) & \cos(j\theta) \end{pmatrix}}_{=:R(-j\theta)} \begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} \tag{2.19}$$

where $R(\alpha)$ is the clock-wise rotation matrix of angle $\alpha$.

**Remark 2.1.** The action of the semi-direct product group $\mathbb{S}^1 \rtimes \mathbb{Z}_2$ on $V_j$ defined by $(\tau_\theta)_{\theta \in \mathbb{S}^1}$ and $\{I, \mathscr{S}\}$ in (2.19) amounts to the action of the orthogonal group $O(2)$, because each $2 \times 2$ orthogonal matrix can be written as the composition of a rotation matrix and a reflection.

Writing $u$ as in (2.10) the quadratic part of the Hamiltonian $\mathcal{H}$ in (1.7) is equal to

$$\mathcal{H}_2(u) := \frac{1}{2} \langle \mathrm{d}\nabla \mathcal{H}(0) u, u \rangle = \frac{1}{2} \sum_{j \neq 0} \Omega_j (\alpha_j^2 + \beta_j^2) + \frac{g \eta_0^2}{2} \tag{2.20}$$

and the momentum

$$\mathcal{I}(u) = \frac{1}{2} \langle \mathrm{d}\nabla \mathcal{I}(0) u, u \rangle = -\frac{1}{2} \sum_{j \neq 0} j (\alpha_j^2 + \beta_j^2)\,. \tag{2.21}$$

**The kernel of $\mathcal{L}_c$.** We now fix some $j_* \in \mathbb{Z} \setminus \{0\}$ and consider the bifurcation speed

$$c_* = \frac{\Omega_{j_*}}{j_*} \in \mathsf{C}\,,$$

cfr. (1.22), (1.20). Clearly, in view (2.16), the subspace $V_{j_*}$ in (2.13) is included in the Kernel of $\mathcal{L}_{c_*}$. It remains to understand if there are other eigenvalues $\mathcal{L}_{c_*}$ equal to zero. In view of the previous analysis

$$\ker \mathcal{L}_{c_*} = \bigoplus_{j \in \mathcal{V}} V_j\,, \quad \mathcal{V} := \big\{ j \in \mathbb{Z} \setminus \{0\}\,,\ \Omega_j = c_* j \big\}\,, \tag{2.22}$$

and the problem is reduced to determine the existence of integers $j$ that solve the equation (1.23). Since $\Omega_j > 0$ for any $j \in \mathbb{Z} \setminus \{0\}$ the integers $j_*$ and $j$ satisfying (1.23) have the same sign

$$\mathrm{sign}(j) = \mathrm{sign}(j_*)\,. \tag{2.23}$$

If there are no other integer $j \neq j_*$ satisfying (1.23) then the Kernel of $\mathcal{L}_{c_*}$ is 2 dimensional (non-resonant case). Otherwise, as we prove in the next section, the Kernel of $\mathcal{L}_{c_*}$ is 4 dimensional (resonant-case).

# 3 The dispersion relation

In this section we prove Proposition 1.1. We study the graph of the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $\xi \mapsto f(\xi) := f(\xi; g, \mathtt{h}, \kappa, \gamma)$ defined, if $\mathtt{h} < +\infty$, by

$$f(\xi) := \frac{\gamma}{2} \frac{\tanh(\mathtt{h}\xi)}{\xi} + \mathrm{sign}(\xi) \sqrt{\Big( g + \kappa \xi^2 + \frac{\gamma^2}{4} \frac{\tanh(\mathtt{h}\xi)}{\xi} \Big) \frac{\tanh(\mathtt{h}\xi)}{\xi}} \tag{3.1}$$

and, in infinite depth $\mathtt{h} = +\infty$, by

$$f(\xi) := \frac{\gamma}{2|\xi|} + \operatorname{sign}(\xi)\sqrt{\Big(g + \kappa\xi^2 + \frac{\gamma^2}{4|\xi|}\Big)\frac{1}{|\xi|}}\,, \tag{3.2}$$

that, for any $j \in \mathbb{Z} \setminus \{0\}$, is equal to the dispersion relation $f(j) := \frac{\Omega_j}{j}$, cfr. (1.21).

Note that $f(\xi; g, \mathtt{h}, \kappa, \gamma) = -f(-\xi; g, \mathtt{h}, \kappa, -\gamma)$ and so it is sufficient to study the function $f(\,\cdot\,; g, \mathtt{h}, \kappa, \gamma)$ for $\gamma \geq 0$.

**Proposition 3.1.** *For any $g \in (0, +\infty)$, $\kappa \in [0, +\infty)$, $\mathtt{h} \in (0, +\infty]$, $\gamma \in [0, +\infty)$, the functions in* (3.1)-(3.2) *satisfy*

$$f(\xi) > 0\,,\ \forall \xi > 0 \qquad \text{and} \qquad f(\xi) < 0\,,\ \forall \xi < 0\,. \tag{3.3}$$

*According to the values of the parameters the following properties hold:*

(1) Finite depth case $\mathtt{h} < +\infty$.

*(1a)* <u>$\kappa > 0$.</u> *Then function $f$ fulfills*

$$\lim_{\xi \to 0^\pm} f(\xi) = \frac{\gamma}{2}\mathtt{h} \pm \sqrt{\Big(g + \frac{\gamma^2}{4}\mathtt{h}\Big)\mathtt{h}}\,, \qquad \lim_{\xi \to \pm\infty} f(\xi) = \pm\infty\,, \quad \lim_{\xi \to 0^\pm} f'(\xi) = 0\,, \tag{3.4}$$

$$\lim_{\xi \to 0^\pm} f''(\xi) = \pm\alpha\Big(B_\pm - \frac{1}{3}\Big) \quad \text{where} \quad \alpha := \frac{2g\mathtt{h}^3}{\sqrt{(4g + \gamma^2\mathtt{h})\mathtt{h}}} \tag{3.5}$$

*and $B_\pm$ are the vorticity-modified Bond numbers in* (1.24)*. Moreover*

$$\begin{aligned}
&\lim_{\xi \to 0^+} f''(\xi; g, \mathtt{h}, \kappa, \gamma) \geq 0 \iff f_{|(0,+\infty)} \text{ is strictly increasing}\,,\\
&\lim_{\xi \to 0^+} f''(\xi; g, \mathtt{h}, \kappa, \gamma) < 0 \iff f_{|(0,+\infty)} \text{ has a unique local minimum}\,,\\
&\lim_{\xi \to 0^-} f''(\xi; g, \mathtt{h}, \kappa, \gamma) \leq 0 \iff f_{|(-\infty,0)} \text{ is strictly increasing}\,,\\
&\lim_{\xi \to 0^-} f''(\xi; g, \mathtt{h}, \kappa, \gamma) > 0 \iff f_{|(-\infty,0)} \text{ has a unique local maximum}\,.
\end{aligned} \tag{3.6}$$

*(1b)* <u>$\kappa = 0$.</u> *The functions $f_{|(0,+\infty)}$ and $f_{|(-\infty,0)}$ are strictly decreasing and*

$$\lim_{\xi \to 0^\pm} f(\xi) = \frac{\gamma}{2}\mathtt{h} \pm \sqrt{\Big(g + \frac{\gamma^2}{4}\mathtt{h}\Big)\mathtt{h}}\,, \quad \lim_{\xi \to \pm\infty} f(\xi) = 0^\pm\,, \quad \lim_{\xi \to 0^\pm} f'(\xi) = 0\,. \tag{3.7}$$

(2) Deep water case $\mathtt{h} = +\infty$.

*(2a)* <u>$\kappa > 0$.</u> *Then $f_{|(0,+\infty)}$ has a unique local minimum, $f_{|(-\infty,0)}$ has a unique local maximum and*

$$\begin{aligned}
&\lim_{\xi \to 0^+} f(\xi) = +\infty\,, \quad \lim_{\xi \to 0^-} f(\xi) = \begin{cases} -\frac{g}{\gamma} & \text{if } \gamma > 0\\ -\infty & \text{if } \gamma = 0\,, \end{cases}\\
&\lim_{\xi \to \pm\infty} f(\xi) = \pm\infty\,, \quad \lim_{\xi \to 0^-} f'(\xi) = \begin{cases} -\frac{g^2}{\gamma^3} & \text{if } \gamma > 0\\ -\infty & \text{if } \gamma = 0\,. \end{cases}
\end{aligned} \tag{3.8}$$

*(2b)* <u>$\kappa = 0$.</u> *The functions $f_{|(0,+\infty)}$ and $f_{|(-\infty,0)}$ are strictly decreasing and*

$$\begin{aligned}
&\lim_{\xi \to 0^+} f(\xi) = +\infty\,, \quad \lim_{\xi \to 0^-} f(\xi) = \begin{cases} -\frac{g}{\gamma} & \text{if } \gamma > 0\\ -\infty & \text{if } \gamma = 0\,, \end{cases}\\
&\lim_{\xi \to \pm\infty} f(\xi) = 0^\pm\,, \quad \lim_{\xi \to 0^-} f'(\xi) = \begin{cases} -\frac{g^2}{\gamma^3} & \text{if } \gamma > 0\\ -\infty & \text{if } \gamma = 0\,. \end{cases}
\end{aligned} \tag{3.9}$$

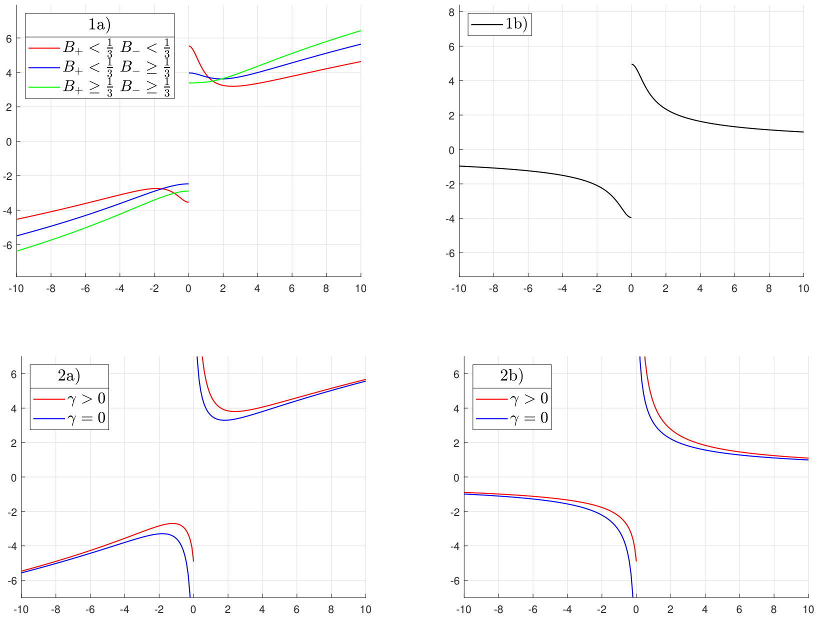


Figure 1: Graphs of $f(\xi)$ according to different values of $(g, \mathtt{h}, \kappa, \gamma)$. The function $f$ has a minimum in $(0, +\infty)$ iff $B_+ < 1/3$ or in case (2a), and have a maximum in $(-\infty, 0)$ iff $B_- < 1/3$ or in case (2a).

Proposition 1.1 is a direct consequence of Proposition 3.1.

*Proof of Proposition 1.1.* In case (1a), by (3.3), (3.4), (3.5), (3.6) the graph of the function $f(\xi)$ has the forms described in Figure 1a) according to the values of $B_\pm$. In particular $f(\xi)$ has a unique absolute minimum for $\xi > 0$ if and only if $B_+ < 1/3$ and a unique absolute maximum for $\xi < 0$ if and only if $B_- < 1/3$. Note that if $\gamma \geq 0$ then $B_+ \leq B_-$, cfr. (1.24), and so the case $B_+ \geq 1/3$ and $B_- < 1/3$ never happens. In the other cases the graphs of the function $f(\xi)$ have the form in Figure 1b), 2a),2b). The existence of integers $j \neq j_*$ such that (1.23) is possible only in cases 1a)-2a). For any $1 \leq j_* < j$ the function $f(j; g, \mathtt{h}, \kappa, \gamma) - f(j_*; g, \mathtt{h}, \kappa, \gamma)$ tends to $+\infty$ as $\kappa \to +\infty$ and tends to $f(j; g, \mathtt{h}, 0, \gamma) - f(j_*; g, \mathtt{h}, 0, \gamma) < 0$ as $\kappa \to 0^+$, which is negative by cases 1b)-2b). Proposition 1.1 follows. $\square$

**Remark 3.2.** The vorticity-modified bond numbers in (1.24) satisfy $\lim_{\mathtt{h} \to +\infty} B_+ = -\infty$ and $B_- = \frac{1}{3} - \frac{g}{3\gamma^2 \mathtt{h}} + O(\mathtt{h}^{-2})$ as $\mathtt{h} \to +\infty$. Thus both $B_+, B_- < \frac{1}{3}$ for sufficiently deep water and the graphs of cases 1a)-2a) look similar. Note also that $-\frac{\mathtt{h}\gamma^2}{6g}\big(1 \pm \big(1 + \frac{4g}{\mathtt{h}\gamma^2}\big)^{\frac{1}{2}}\big) < 1/3$ for any $g, \mathtt{h}, \gamma$.

The rest of this section is dedicated to prove Proposition 3.1.

Property (3.3) directly follows by (3.1)-(3.2). The function $f(\xi)$ in (3.1) admits an analytic extension at $\xi = 0$ and a Taylor expansion provides the limits of $f(\xi)$ as $\xi \to 0^\pm$ in (3.4), (3.7) and (3.5). The limits of $f(\xi)$ for $\xi \to \pm\infty$ follow directly. Also the limits (3.8)-(3.9) of the function $f(\xi)$ in (3.2) are directly verified.

In order to prove (3.6), the fact that in case (2a) the function $f_{|(0,+\infty)}(\xi)$ has a unique local minimum and that in cases (1b)-(2b) the function $f_{|(0,+\infty)}(\xi)$ is strictly decreasing (with the analogous properties for $f_{|(-\infty,0)}(\xi)$) we first provide the following lemma, whose proof is related to [30]. We denote

$$f_+(\xi) := f_{|(0,+\infty)}(\xi), \quad f_-(\xi) := f_{|(-\infty,0)}(-\xi), \quad \forall \xi > 0. \tag{3.10}$$

**Lemma 3.3.** *If $\kappa = 0$ the functions $f_\pm(\xi)$ have no critical points. For any value of the parameters $(g, \mathtt{h}, \kappa, \gamma)$ with $\kappa > 0$, any critical point $\overline{\xi} > 0$ of $f_+(\xi)$, resp. $f_-(\xi)$, is a strict local minimum, resp. maximum, i.e.*

$$f'_\pm(\overline{\xi}) = 0 \quad \Longrightarrow \quad \pm f''_\pm(\overline{\xi}) > 0\,. \tag{3.11}$$

*The function $f_+(\xi)$, resp. $f_-(\xi)$, has at most one strict local minimum, resp. maximum, in $[0, +\infty)$.*

*Proof.* We prove the deep water case $\mathtt{h} = +\infty$. The case $\mathtt{h} < +\infty$ is analogous and is covered in [30]. In view of (3.2) we have, for any $\xi > 0$,

$$f_+(\xi) + f_-(\xi) = \frac{\gamma}{\xi}\,, \quad f_+(\xi) f_-(\xi) = -\frac{g + \kappa\xi^2}{\xi}\,, \tag{3.12}$$

and thus $f_\pm(\xi)$ solves the second order equation

$$\xi f_\pm(\xi)^2 - \gamma f_\pm(\xi) - (g + \kappa\xi^2) = 0\,, \quad \forall \xi > 0\,. \tag{3.13}$$

The reciprocal functions

$$g_+(\xi) := \frac{1}{f_+(\xi)} > 0\,, \quad g_-(\xi) := \frac{1}{f_-(\xi)} < 0\,, \quad \forall \xi > 0\,, \tag{3.14}$$

are well defined since $f_+(\xi) > 0$ and $f_-(\xi) < 0$ for any $\xi > 0$ by (3.10), (3.3), and solve, in view of (3.13),

$$(g + \kappa\xi^2) g_\pm^2(\xi) + \gamma g_\pm(\xi) = \xi\,, \quad \forall\, \xi > 0\,. \tag{3.15}$$

The critical points of $g_\pm$ are critical points of $f_\pm$ and viceversa. Differentiating (3.15) in $\xi$ we get, for any $\xi > 0$,

$$2\kappa\xi g_\pm^2(\xi) + 2(g + \kappa\xi^2) g_\pm(\xi) g'_\pm(\xi) + \gamma g'_\pm(\xi) = 1\,. \tag{3.16}$$

For $\kappa = 0$ the function $f_\pm(\xi)$ has no critical points. Indeed, if $f'_\pm(\overline{\xi}) = 0$ then $g'_\pm(\overline{\xi}) = 0$ and by (3.16) we get $0 = 1$. This contradiction proves the first statement of the lemma.

Thus in the sequel we assume $\kappa > 0$.

Differentiating (3.16) at $\xi = \overline{\xi}$ we get $\big(2(g + \kappa\overline{\xi}^2) g_\pm(\overline{\xi}) + \gamma\big) g''_\pm(\overline{\xi}) = -2\kappa g_\pm^2(\overline{\xi})$ and using (3.15) we get

$$\Big((g + \kappa\overline{\xi}^2) g_\pm(\overline{\xi}) + \frac{\overline{\xi}}{g_\pm(\overline{\xi})}\Big) g''_\pm(\overline{\xi}) = -2\kappa g_\pm^2(\overline{\xi})\,. \tag{3.17}$$

Since $g_+(\overline{\xi}) > 0$ (cfr. (3.14)) we deduce by (3.17) that $g''_+(\overline{\xi}) < 0$ and then $f''_+(\overline{\xi}) = -\frac{g''_+(\overline{\xi})}{g_+^2(\overline{\xi})} > 0$ and thus $f''_+(\overline{\xi}) > 0$. Similarly, since $g_-(\overline{\xi}) < 0$ we deduce $g''_-(\overline{\xi}) > 0$ and thus $f''_-(\overline{\xi}) < 0$. This proves (3.11).

Let us prove the last claim of the lemma. If $f_+(\xi)$ has two distinct local minima $0 \le \xi_1 < \xi_2$, then $f_+(\xi)$ has a local maximum point in $(\xi_1, \xi_2)$. This is a contradiction with (3.11). The claim for $f_-(\xi)$ follows similarly. □

The next lemma allows to deduce that a 4 dimensional Kernel of $\mathcal{L}_{c_*}$ may occur if and only if $B_\pm < 1/3$.

**Lemma 3.4.** (3.6) *holds.*

*Proof.* We prove the first equivalence in (3.6). The other ones are analogous. Lemma 3.3 implies that $f_+(\cdot; g, \mathtt{h}, \kappa, \gamma)$ is increasing in $[0, +\infty)$ if and only if it is *strictly* increasing. Therefore (3.6) is equivalent to the following claim.

**Claim:** $\lim_{\xi\to 0^+} f''_+(\xi; g, \mathtt{h}, \kappa, \gamma) \geq 0$ *if and only if* $\xi \mapsto f_+(\xi; g, \mathtt{h}, \kappa, \gamma)$ *is increasing.*

Since $\lim_{\xi\to 0^+} f'_+(\xi; g, \mathtt{h}, \kappa, \gamma) = 0$ by (3.4), the implication $\Leftarrow$ is trivial. To prove the other implication we consider two different cases:

*Case 1:* $\lim_{\xi\to 0^+} f''_+(\xi; g, \mathtt{h}, \kappa, \gamma) > 0$. Then $f'_+(\xi; g, \mathtt{h}, \kappa, \gamma) > 0$ for any $\xi > 0$ by Lemma 3.3. Thus $\xi \mapsto f_+(\xi; g, \mathtt{h}, \kappa, \gamma)$ is increasing.

*Case 2:* $\lim_{\xi\to 0^+} f''_+(\xi; g, \mathtt{h}, \kappa, \gamma) = 0$. By (3.5) and (1.24) we have

$$0 = \lim_{\xi\to 0^+} f''_+(\xi; g, \mathtt{h}, \kappa, \gamma) = \alpha\Big(\frac{\kappa}{g\mathtt{h}^2} - \frac{\gamma}{6g}\sqrt{(4g+\gamma^2\mathtt{h})\mathtt{h}} - \frac{\mathtt{h}\gamma^2}{6g} - \frac{1}{3}\Big) \quad \text{with} \quad \alpha > 0\,.$$

Therefore, for any $\varepsilon > 0$, $\lim_{\xi\to 0^+} f''(\xi; g, \mathtt{h}, \kappa+\varepsilon, \gamma) > 0$ and thus, by case 1, the function $\xi \mapsto f_+(\xi; g, \mathtt{h}, \kappa+\varepsilon, \gamma)$ is increasing in $\xi \in [0, +\infty)$. Therefore the limit function $f_+(\xi; g, \mathtt{h}, \kappa, \gamma)$ as $\varepsilon \to 0^+$ is increasing as well in $\xi \in [0, +\infty)$. The claim is proved. ☐

By Lemma 3.3 in case (2a) the function $f_{|(0,+\infty)}(\xi)$ has a unique local minimum. In cases (1b)-(2b) the function $f_{|(0,+\infty)}(\xi)$ is strictly decreasing still by Lemma 3.3. The proof of Proposition 3.1 is complete.

## 4 Variational Lyapunov-Schmidt reduction

We decompose the phase space $L^2 \times L^2 := L^2(\mathbb{T}, \mathbb{R}) \times L^2(\mathbb{T}, \mathbb{R})$ equipped with the symplectic form $\boldsymbol{\mathcal{W}}$ in (1.8) as

$$L^2 \times L^2 = V \oplus W \tag{4.1}$$

where, recalling (2.1), (2.2) and (2.22),

$$V := \ker \mathcal{L}_{c_*} = \big\{c_*\partial_x v + J\mathrm{d}\nabla\mathcal{H}(0)v = 0\big\} = \big\{\mathrm{d}\nabla(\mathcal{H} + c_*\mathcal{I})(0)v = 0\big\} \tag{4.2}$$

$$= \bigoplus_{j\in\mathcal{V}} V_j\,, \tag{4.3}$$

$$W := V^{\perp_{\boldsymbol{\mathcal{W}}}} := V_0 \oplus \mathscr{W}\,, \qquad \mathscr{W} := \overline{\bigoplus_{j\in\mathbb{Z}\setminus\{0\}, j\notin\mathcal{V}} V_j}^{L^2\times L^2}\,, \tag{4.4}$$

and $V_j$, $j \in \mathbb{Z}$, are the bi-dimensional symplectic subspaces defined in (2.13). The subspaces $V$ and $W$ are symplectic and each one is the symplectic orthogonal of the other. We denote by $\Pi_V$ and $\Pi_W$ the *symplectic* projectors on $V$, respectively $W$, induced by the decomposition (4.1). Since $V$ and $W$ are symplectic orthogonal, the projectors $\Pi_V$, $\Pi_W$ satisfy

$$\boldsymbol{\mathcal{W}}(\Pi_V u, u_1) = \boldsymbol{\mathcal{W}}(u, \Pi_V u_1)\,, \quad \boldsymbol{\mathcal{W}}(\Pi_W u, u_1) = \boldsymbol{\mathcal{W}}(u, \Pi_W u_1)\,, \quad \forall u, u_1 \in L^2 \times L^2\,. \tag{4.5}$$

In order to solve (1.15) we implement a symplectic Lyapunov Schmidt-reduction. According to (4.1) the space $X$ defined in (1.18) and the target space $Y$ defined in (1.19) admit the decomposition

$$X = V \oplus (W \cap X)\,, \quad Y = V \oplus (W \cap Y)\,, \tag{4.6}$$

in symplectic orthogonal subspaces. We denote by $\Pi_{W\cap X}$ and $\Pi_{W\cap Y}$ the symplectic projectors on $W \cap X$ and $W \cap Y$ induced by (4.6). Decomposing uniquely $u \in X$ as $u = v + w$ with $v \in V$ and $w \in W \cap X$, the equation (1.15) is then equivalent to the system

$$\begin{cases} \Pi_V \mathcal{F}(c, v + w) = 0 \\ \Pi_{W\cap Y}\mathcal{F}(c, v + w) = 0\,. \end{cases} \tag{4.7}$$

We call the first equation the bifurcation equation and the second one the range equation. We now solve the range equation by means of the implicit function theorem.

We denote by $B_r^V(0)$ the ball of radius $r$ and center 0 in $V$.

**Lemma 4.1. (Solution of range equation)** *There exists an analytic function* $w : B_r(c_*) \times B_r^V(0) \subset \mathbb{R} \times V \to W \cap X$ *defined in a neighborhood of* $(c_*, 0)$ *satisfying*

$$w(c,0) = 0\,, \qquad \mathrm{d}_v w(c,0) = 0\,, \quad \forall c \in B_r(c_*)\,, \tag{4.8}$$

*such that*

$$\Pi_{W\cap Y}\mathcal{F}(c, v + w(c,v)) = 0\,. \tag{4.9}$$

*The function* $w(c,\cdot)$ *is equivariant with respect to the involution* $\mathscr{S}$ *and the translations* $\tau_\theta$*, namely*

$$w(c, \mathscr{S}v) = \mathscr{S}w(c,v)\,, \quad w(c, \tau_\theta v) = \tau_\theta w(c,v)\,, \quad \forall v \in B_r^V(0)\,, \quad \forall \theta \in \mathbb{R}\,. \tag{4.10}$$

*Proof.* We apply the implicit function theorem to

$$G : \mathbb{R} \times V \times (W \cap X) \to (W \cap Y)\,, \ (c, v, w) \mapsto G(c,v,w) := \Pi_{W\cap Y}\mathcal{F}(c, v + w)\,.$$

The map $G$ is analytic in a small neighborhood of $(c_*, 0, 0)$ using Theorem 1.2 in [6] about the analyticity of the Dirichlet-Neumann operator and the algebra properties of $H^{\sigma,s}(\mathbb{T})$. Theorem 1.2 of [6] is proved in the more delicate deep water case $\mathtt{h} = +\infty$ but its proof also holds if $\mathtt{h} < +\infty$, in this case see also [34]. It results $G(c,0,0) = 0$ and we claim that

$$\mathrm{d}_w G(c_*, 0, 0) = \Pi_{W\cap Y}\mathrm{d}_u\mathcal{F}(c_*, 0) = \Pi_{W\cap Y}\mathcal{L}_{c_*} : (W \cap X) \to W \cap Y \tag{4.11}$$

is an isomorphism. Indeed, in view of (2.15), (2.14), (2.17) and recalling (4.3) there exists a formal inverse $\mathcal{A} : \langle v_0^{(2)} \rangle \oplus \mathscr{W} \to \langle v_0^{(1)} \rangle \oplus \mathscr{W}$ of $\Pi_{W\cap Y}\mathcal{L}_{c_*}$, where we $v_0^{(1)}, v_0^{(2)}$ are in (2.11), defined by

$$\mathcal{A}v_0^{(2)} := -\frac{1}{g}v_0^{(1)}\,, \quad \mathcal{A} := \mathcal{M}\mathcal{C}\begin{pmatrix} (c_*\partial_x - \mathrm{i}\Omega(D))^{-1} & 0 \\ 0 & (c_*\partial_x + \mathrm{i}\overline{\Omega(D)})^{-1} \end{pmatrix} \mathcal{C}^{-1}\mathcal{M}^{-1} \quad \text{on } \mathscr{W}\,.$$

By (2.3) and (2.4) the operator $\mathcal{A}$ on $\mathscr{W}$ is given by the matrix of Fourier multipliers

$$\begin{pmatrix} A_{1,1}(D) & A_{1,2}(D) \\ A_{2,1}(D) & A_{2,2}(D) \end{pmatrix} \tag{4.12}$$

whose symbols are (recall that the symbol of $\overline{\Omega(D)}$ is $\Omega_{-\xi}$)

$$A_{1,1}(\xi) = A_{2,2}(\xi) = \frac{1}{2}\frac{2c_*\xi + \Omega_{-\xi} - \Omega_\xi}{(c_*\xi - \Omega_\xi)(c_*\xi + \Omega_{-\xi})}\,, \tag{4.13}$$

$$A_{1,2}(\xi) = \frac{1}{2}M_\xi^2\frac{\Omega_\xi + \Omega_{-\xi}}{(c_*\xi - \Omega_\xi)(c_*\xi + \Omega_{-\xi})}\,, \ A_{2,1}(\xi) = \frac{1}{2}M_\xi^{-2}\frac{\Omega_\xi + \Omega_{-\xi}}{(c_*\xi - \Omega_\xi)(c_*\xi + \Omega_{-\xi})}\,. \tag{4.14}$$

In view of (2.3) the symbol $M_\xi$ has order $-\frac{1}{4}$ if $\kappa > 0$ and order $\frac{1}{4}$ if $\kappa = 0$. Estimating the orders of the symbols (4.13)-(4.14) using (1.21) and (2.7), the operator $\mathcal{A}$ in (4.12) is a matrix of Fourier multipliers as

$$\begin{pmatrix} OPS^{-2} & OPS^{-2} \\ OPS^{-1} & OPS^{-2} \end{pmatrix} \quad \text{for } \kappa > 0 \quad \text{and} \quad \begin{pmatrix} OPS^{-1} & OPS^{-1} \\ OPS^{-2} & OPS^{-1} \end{pmatrix} \quad \text{for } \kappa = 0\,.$$

Note that for $\kappa > 0$ the symbol $A_{1,1}(\xi) = A_{2,2}(\xi)$ has order $-2$, because, by the cancellation (2.7), the symbol $\Omega_\xi - \Omega_{-\xi}$ has order 0. In conclusion in both cases $\mathcal{A}$ can be extended to an operator from $W \cap Y = (\mathscr{W} \oplus \langle v_0^{(2)} \rangle) \cap Y$ to $W \cap X = (\langle v_0^{(1)} \rangle \oplus \mathscr{W}) \cap X$. This proves that $\mathrm{d}_w G(c_*, 0, 0)$ in (4.11) is an isomorphism.

The existence of a unique analytic solution $w(c, v)$ of (4.9) defined for $(c, v)$ close to $(c_*, 0)$ follows by the analytic implicit function theorem. Since $G(c, 0, 0) = 0$ by uniqueness we have $w(c, 0) = 0$ for any $c \in B_r(c_*)$. Next we compute the derivative of $w(c, v)$ with respect to $v$. For any $c$ close to $c_*$ the differential $\mathrm{d}_w G(c, 0, 0)$ is invertible as well as $\mathrm{d}_w G(c_*, 0, 0)$ and

$$\mathrm{d}_v w(c, 0) = -\mathrm{d}_w G(c, 0, 0)^{-1} \Pi_{W \cap Y} \mathcal{L}_{c|V} = 0$$

because $\mathcal{L}_c V \subset V$ for any $c$, cfr. (2.14).

Finally in order to prove (4.10) note that by (1.10), (1.11) and (1.14),

$$\Pi_{W \cap Y} \mathcal{F}(c, \tau_\theta u) = \tau_\theta \Pi_{W \cap Y} \mathcal{F}(c, u) \,, \quad \Pi_{W \cap Y} \mathcal{F}(c, \mathscr{S} u) = -\mathscr{S} \Pi_{W \cap Y} \mathcal{F}(c, u)$$

(use also $\mathscr{S}^\top = \mathscr{S} = \mathscr{S}^{-1}$). Therefore by uniqueness we deduce (4.10). □

In view of the previous lemma the system (4.7) reduces to solve the bifurcation equation

$$\Pi_V \mathcal{F}(c, v + w(c, v)) = 0 \tag{4.15}$$

where $w(c, v)$ is the solution of the range equation. The equation (4.15) is still variational since $\Pi_V \mathcal{F}(c, v + w(c, v))$ is the symplectic gradient of the "reduced Hamiltonian" $\Phi(c, \cdot)$ defined below.

**Lemma 4.2. (Variational structure of the bifurcation equation)** *The function* $\Phi(\cdot, \cdot) : B_r(c_*) \times B_r^V(0) \subset \mathbb{R} \times V \to \mathbb{R}$, $(c, v) \mapsto \Phi(c, v)$, *defined by*

$$\Phi(c, v) := \Psi(c, v + w(c, v)) = (\mathcal{H} + c\mathcal{I})(v + w(c, v)) \,, \tag{4.16}$$

*is analytic on* $B_r(c_*) \times B_r^V(0)$, *and satisfies, for any* $(c, v) \in B_r(c_*) \times B_r^V(0)$,

$$\Phi(c, \mathscr{S} v) = \Phi(c, v) \,, \quad \Phi(c, \tau_\theta v) = \Phi(c, v) \,, \quad \forall \theta \in \mathbb{R} \,, \tag{4.17}$$

*and*

$$\mathrm{d}_v \Phi(c, v)[\widehat{v}] = \mathrm{d}_u \Psi(c, v + w(c, v))[\widehat{v}] = \boldsymbol{\mathcal{W}}(\Pi_V \mathcal{F}(c, v + w(c, v)), \widehat{v}) \,, \quad \forall \widehat{v} \in V \,. \tag{4.18}$$

*Therefore if* $\overline{v} \in B_r^V(0)$ *is a critical point of* $v \mapsto \Phi(c, v)$ *then* $\overline{u} := \overline{v} + w(c, \overline{v})$ *is a solution of* $\mathcal{F}(c, u) = 0$.

*Proof.* We first note that the range equation (4.9) has a variational meaning, being $w(c, v)$ a critical point of the functional $w \mapsto \Psi(c, v + w)$. Indeed for any $\widehat{w} \in W$ we have, recalling (1.9), (1.15),

$$\begin{aligned} \mathrm{d}_u \Psi(c, v + w(c, v))[\widehat{w}] &= \boldsymbol{\mathcal{W}}(\mathcal{F}(c, v + w(c, v)), \widehat{w}) \\ &= \boldsymbol{\mathcal{W}}(\underbrace{\Pi_W \mathcal{F}(c, v + w(c, v))}_{=0 \text{ by } (4.9)}, \widehat{w}) = 0 \,. \end{aligned} \tag{4.19}$$

Then, differentiating (4.16) with respect to $v \in V$ in the direction $\widehat{v} \in V$, we get

$$\begin{aligned} \mathrm{d}_v \Phi(c, v)[\widehat{v}] &= \mathrm{d}_u \Psi(c, v + w(c, v))[\widehat{v}] + \underbrace{\mathrm{d}_u \Psi(c, v + w(c, v))[\mathrm{d}_v w(c, v)[\widehat{v}]]}_{=0 \text{ by } (4.19) \text{ and } \mathrm{d}_v w(c,v)[\widehat{v}] \in W} \\ &= \boldsymbol{\mathcal{W}}(\mathcal{F}(c, v + w(c, v)), \widehat{v}) = \boldsymbol{\mathcal{W}}(\Pi_V \mathcal{F}(c, v + w(c, v)), \widehat{v}) \end{aligned} \tag{4.20}$$

proving (4.18). In conclusion, by (4.20), if $\overline{v} \in B_r^V(0)$ is a critical point of $v \mapsto \Phi(c, v)$, then $\Pi_V \mathcal{F}(c, \overline{v} + w(c, \overline{v})) = 0$ because $\boldsymbol{\mathcal{W}}$ is non-degenerate. □

Summarizing, we have proved that the original problem (1.15) is equivalent, locally near $(c_*, 0)$, to find critical points of the functional $\Phi(c, \cdot)$ for some value of $c$.

In order to prove the existence of non-trivial critical points of $\Phi(c, v)$ in Sections 5.1 and 5.2 we first expand it close to $v = 0$.

**Lemma 4.3.** *For any $(c, v) \in B_r(c_*) \times B_r^V(0)$ the function $\Phi(c, v)$ in (4.16) has the form*

$$\Phi(c, v) = (c - c_*)\mathcal{I}(v) + G_{\geq 3}(c, v) = \Big(1 - \frac{c}{c_*}\Big)\mathcal{H}_2(v) + G_{\geq 3}(c, v) \tag{4.21}$$

$$= \frac{1}{2}\boldsymbol{\mathcal{W}}(\mathcal{L}_c v, v) + G_{\geq 3}(c, v) \tag{4.22}$$

*where $\mathcal{H}_2(v) := \frac{1}{2}\langle \mathrm{d}\nabla\mathcal{H}(0)v, v\rangle$, the speed $c_* \neq 0$ is defined in (1.22), and $G_{\geq 3}(c, v)$ is an analytic function vanishing at $v = 0$ with cubic order for any $c \in B_r(c_*)$.*

*Proof.* A Taylor expansion of (4.16), using $\mathcal{H}(0) = \mathcal{I}(0) = 0$, $\nabla\mathcal{H}(0) = \nabla\mathcal{I}(0) = 0$ and (4.8) gives

$$\begin{aligned}\Phi(c, v) &= \frac{1}{2}\big\langle \mathrm{d}\nabla(\mathcal{H} + c\mathcal{I})(0)v, v\big\rangle + G_{\geq 3}(c, v) \\ &\overset{(1.8),(1.15)}{=} \frac{1}{2}\boldsymbol{\mathcal{W}}\big(\underbrace{\mathrm{d}\mathcal{F}(c, 0)}_{=\mathcal{L}_c \text{ by } (2.1)} v, v\big) + G_{\geq 3}(c, v)\,.\end{aligned} \tag{4.23}$$

This proves (4.22). Formula (4.21) follows by (4.23) because, by (4.2) and (2.21),

$$\mathcal{H}_2(v) = \frac{1}{2}\langle \mathrm{d}\nabla\mathcal{H}(0)v, v\rangle = -c_*\mathcal{I}(v)\,, \quad \mathcal{I}(v) = \frac{1}{2}\langle \mathrm{d}\nabla\mathcal{I}(0)v, v\rangle$$

for any $v \in V$. ☐

Recalling (4.3) and (2.10) any function of $V = \ker \mathcal{L}_{c_*}$ in (2.22) can be written as

$$v = \sum_{j\in\mathcal{V}} \alpha_j v_j^{(1)} + \beta_j v_j^{(2)} \quad \text{where } \mathcal{V} \text{ is defined in (2.22)}\,, \tag{4.24}$$

the vectors $(v_j^{(1)}, v_j^{(2)})_{j\in\mathcal{V}}$ defined in (2.11) form a symplectic basis of $V$, and the coordinates $\alpha_j := \alpha_j(v)$, $\beta_j := \beta_j(v)$ are given in (2.12). In view of (2.20), (2.21), we have

$$\mathcal{I}(v) = -\frac{1}{2}\sum_{j\in\mathcal{V}} j(\alpha_j^2 + \beta_j^2)\,, \quad \mathcal{H}_2(v) = \frac{1}{2}\sum_{j\in\mathcal{V}} \Omega_j(\alpha_j^2 + \beta_j^2)\,. \tag{4.25}$$

**Remark 4.4.** The cubic Hamiltonian $G_{\geq 3}(c, v)$ in (4.21) is in "Birkhoff resonant normal form", namely $\{G_{\geq 3}(c, \cdot), \mathcal{I}\} = 0$ where $\{F, G\} := \boldsymbol{\mathcal{W}}(X_F, X_G)$ is the Poisson bracket between two functions on $V$. Indeed the reduced Hamiltonian $\Phi(c, \cdot)$ in (4.16) defined on the symplectic space $V$ in (4.3) has the prime integral $\mathcal{I}(v)$, namely $\{\Phi(c, \cdot), \mathcal{I}\} = 0$.

**Notation.** In the sequel we denote $\Phi(c, v)$ equivalently as $\Phi(c, (\alpha_j, \beta_j)_{j\in\mathcal{V}})$.

By Proposition 1.1 the space $V$ can be either 2 or 4 dimensional.

Non resonant case. If $\ker \mathcal{L}_{c_*} = V = \{v = \alpha_* v^{(1)}_{j_*} + \beta_* v^{(2)}_{j_*}\}$ is 2-dimensional (for simplicity we denote $\alpha_* = \alpha_{j_*}$ and $\beta_* = \beta_{j_*}$), the symmetries (4.17) show, recalling (2.19), that

$$\Phi(c, \alpha_*, \beta_*) = \Phi(c, R(-j_*\theta)(\alpha_*, \beta_*)) \,, \ \forall \theta \in \mathbb{R} \,, \quad \Phi(c, \alpha_*, \beta_*) = \Phi(c, \alpha_*, -\beta_*) \,,$$

and therefore the functional $\Phi(c, \alpha_*, \beta_*)$ is a radial (i.e. is a function of $\alpha_*^2 + \beta_*^2$) for any $c$. This shows that all the critical points of $\Phi(c, \alpha_*, \beta_*)$ are obtained by rotations of critical points of the function $\alpha_* \mapsto \Phi(c, \alpha_*, 0)$ of one variable only. In view of (4.21) and (4.25), we have

$$\Phi(c, \alpha_*, 0) = -\tfrac{1}{2}(c - c_*) j_* \alpha_*^2 + G_{\geq 3}(c, \alpha_*, 0) \,. \tag{4.26}$$

Since

$$\partial_c \partial^2_{\alpha_*} \Phi(c, \alpha_*, 0) = -j_* \neq 0 \,, \tag{4.27}$$

by the implicit function theorem, for any $\alpha_*$ small enough, there exists a unique speed $c(\alpha_*)$, analytic in $\alpha_*$, such that $(\partial_{\alpha_*}\Phi)(c(\alpha_*), \alpha_*) = 0$. Actually (4.27) is nothing but the Crandall-Rabinowitz transversality condition, which requires $\partial_c \mathcal{L}_c v^{(1)}_{j_*}|_{c_*}$ to not belong to the range of

$$\mathcal{L}_{c_*} : \big(\langle v^{(1)}_0 \rangle \bigoplus_{j \in \mathbb{Z}\setminus\{0\}} \langle v^{(1)}_j \rangle\big) \cap X \to \big(\langle v^{(2)}_0 \rangle \bigoplus_{j \in \mathbb{Z}\setminus\{0\}} \langle v^{(2)}_j \rangle\big) \cap Y =: Y^{(2)} \,.$$

Here the domain and target spaces have been restricted so that $\ker \mathcal{L}_{c_*}$ is 1-dimensional and the range $\mathcal{R} := \big(\langle v^{(2)}_0 \rangle \bigoplus_{j \neq j_*} \langle v^{(2)}_j \rangle\big) \cap Y$ is of codimension 1. Now, in view of (4.22),

$$\partial_c \partial^2_{\alpha_*} \Phi(c, \alpha_*, 0) = \boldsymbol{\mathcal{W}}\big(\partial_c \mathcal{L}_c v^{(1)}_{j_*}, v^{(1)}_{j_*}\big) \,,$$

which is non-zero iff $\partial_c \mathcal{L}_c v^{(1)}_{j_*} \notin \langle v^{(1)}_{j_*} \rangle^{\perp_{\boldsymbol{\mathcal{W}}}} \cap Y^{(2)} \equiv \mathcal{R}$, namely the transversality condition.

All the non-trivial solutions of (1.15) can be parametrized as rotations of the Stokes waves

$$u_\epsilon = \epsilon v^{(1)}_{j_*} + \underbrace{w\big(c_\epsilon, \epsilon v^{(1)}_{j_*}\big)}_{=\mathcal{O}(\epsilon^2)} \,, \quad c_\epsilon = c_* + \underbrace{\widetilde{c}_\epsilon}_{=\mathcal{O}(\epsilon^2)} \,, \quad v^{(1)}_{j_*} := \begin{pmatrix} M_{j_*} \cos(j_* x) \\ M_{j_*}^{-1} \sin(j_* x) \end{pmatrix} \,.$$

The Stokes wave $\epsilon \to u_\epsilon$ is analytic in $\epsilon$ and $x$. This follows from the analyticity of the solution of the range equation proved in Lemma 4.1. The function $w\big(c_\epsilon, \epsilon v^{(1)}_{j_*}\big)$ has the first component even in $x$ and the second one odd in $x$, because $w(c_\epsilon, \epsilon v^{(1)}_{j_*}) = \mathscr{S} w(c_\epsilon, \epsilon v^{(1)}_{j_*})$ by (4.10) and $\mathscr{S} v^{(1)}_{j_*} = v^{(1)}_{j_*}$.

In other words, in the non-resonant case when $\dim \ker \mathcal{L}_{c_*} = 2$, any Stokes wave is a rotation of a Stokes wave with $\eta(x)$ even and $\zeta(x)$ odd and the applicability of the Crandall-Rabinowitz bifucation theorem is an automatic consequence of the Hamiltonian variational structure of the equations.

## 5 Resonant case

We now consider the resonant case when $\dim \ker \mathcal{L}_{c_*} = 4$. In view of (2.22), (2.13) we have

$$\ker \mathcal{L}_{c_*} = V = \Big\{ v = \alpha_{j_*} v^{(1)}_{j_*} + \beta_{j_*} v^{(2)}_{j_*} + \alpha_j v^{(1)}_j + \beta_j v^{(2)}_j \colon \alpha_{j_*}, \alpha_j, \beta_{j*}, \beta_j \in \mathbb{R} \Big\}$$

where $j \neq j_*$ is the other integer such that $\frac{\Omega_j}{j} = c_* = \frac{\Omega_{j_*}}{j_*}$ (cfr. (1.23)).

Note that, by (4.17) and (2.19), the function $\Phi(c,v)$ in (4.16) satisfies the symmetries

$$\Phi(c,\alpha_{j_*},\beta_{j_*},\alpha_j,\beta_j) = \Phi\big(c, R(-j_*\theta)(\alpha_{j_*},\beta_{j_*}), R(-j\theta)(\alpha_j,\beta_j)\big)\,,\ \forall\theta\in\mathbb{R}\,, \tag{5.1}$$

$$\Phi(c,\alpha_{j_*},\beta_{j_*},\alpha_j,\beta_j) = \Phi(c,\alpha_{j_*},-\beta_{j_*},\alpha_j,-\beta_j)\,. \tag{5.2}$$

The reversibility symmetry (5.2) implies that the derivatives $(\partial_{\beta_j}\Phi)(c,\alpha_{j_*},0,\alpha_j,0) = 0 = (\partial_{\beta_{j_*}}\Phi)(c,\alpha_{j_*},0,\alpha_j,0)$ and thus a critical point of

$$(\alpha_{j_*},\alpha_j) \mapsto \Phi(c,\alpha_{j_*},0,\alpha_j,0) \tag{5.3}$$

is also a critical point of $(\alpha_{j_*},\beta_{j_*},\alpha_j,\beta_j) \mapsto \Phi(c,\alpha_{j_*},\beta_{j_*},\alpha_j,\beta_j)$. The corresponding Stokes wave $u = \alpha_{j_*}v^{(1)}_{j_*} + \alpha_j v^{(1)}_j + w(c,\alpha_{j_*}v^{(1)}_{j_*} + \alpha_j v^{(1)}_j)$ has the $\eta$ component which is *even*. The Stokes waves in the orbit $\{\tau_\theta u\}_{\theta\in\mathbb{R}}$ are called *symmetric*, cfr. [29], [41]. However (5.1) does not imply that all the Stokes waves of (1.7) are symmetric, as shown in [29], [41]. Equivalently there could be critical points of $v\mapsto\Phi(c,v)$ which are not obtained by a $\theta$-translation of critical points of (5.3).

**Definition 5.1. (Geometrically distinct critical points)** *Two non-trivial critical points of $\Phi(c,v)$ are geometrically distinct if they are not obtained by applying the translation operator $\tau_\theta$ or the reflection operator $\mathscr{S}$ to the other one. Equivalently if they are not in the same orbit generated by the action of $\mathbb{S}^1 \rtimes \mathbb{Z}_2 \cong O(2)$, cfr. Remark 2.1.*

We are going to prove the existence of non trivial critical orbits of $\Phi(c,v)$, parametrized by the speed $c\sim c_*$ in Section 5.1, or the momentum $\mathcal{I}(v) = a \sim 0$, in Section 5.2.

## 5.1 Stokes waves parametrized by the speed

In this section we prove Theorem 1.2. For definiteness in the sequel we assume that $j_*, j < 0$ (recall that $j, j_*$ have the *same* sign by (2.23)). The other case follows similarly. With this choice the momentum in (4.25) is the positive definite quadratic form

$$\mathcal{I}(v) = \underbrace{\frac{1}{2}|j_*|(\alpha_{j_*}^2+\beta_{j_*}^2) + \frac{1}{2}|j|(\alpha_j^2+\beta_j^2)}_{=:\|v\|_*^2}\,, \tag{5.4}$$

and $\|v\|_* := \mathcal{I}(v)^{1/2}$ is a norm on $V$. Thus the functional $\Phi(c,v)$ has, by (4.21), a local minimum at $v=0$ for any $c>c_*$, and a local maximum for any $c<c_*$.

For simplicity of notation we denote $B^V_r(0) \equiv B^V_r$.

Consider the analytic function $\Phi(c_*,v) = G_{\geq 3}(c_*,v)$ which vanishes cubically at $v=0$ by Lemma 4.3. If $v=0$ is *not* an isolated critical point of $\Phi(c_*,\cdot)$ then alternative $(i)$ of Theorem 1.2 holds: there exists a sequence $v_n\to 0$ of critical points of $\Phi(c_*,\cdot)$ and thus, in view of Lemma 4.2, a sequence of solutions

$$u_n = v_n + w(c_*,v_n) \quad \text{of} \quad \mathcal{F}(c_*,u_n) = 0\,, \quad \text{with} \quad v_n\to 0\,.$$

Thus, in the following, we assume $v=0$ is an isolated critical point of $\Phi(c_*,\cdot)$. Consequently $v=0$ is either

$(a)$ a strict local maximum or minimum for $\Phi(c_*,\cdot)$ ;

$(b)$ $\Phi(c_*,\cdot)$ takes on both positive and negative values near $v=0$ .

Case $(a)$ leads to alternative $(ii)$ and Case $(b)$ leads to alternative $(iii)$ of Theorem 1.2.

*Case (a)*: Suppose $v = 0$ is a strict local maximum of $\Phi(c_*, \cdot)$ (to handle the case of a strict local minimum just replace $\Phi$ with $-\Phi$). Since $\Phi(c_*, 0) = 0$, for $r > 0$ small enough,

$$\exists \beta > 0 \quad \text{such that} \quad \Phi_{|\partial B_r^V}(c_*, \cdot) \leq -2\beta \,.$$

By continuity, for $c$ sufficiently close to $c_*$,

$$\Phi_{|\partial B_r^V}(c, \cdot) \leq -\beta \,. \tag{5.5}$$

By (4.21) and (5.4), for any $c > c_*$, the functional $\Phi(c, \cdot)$ has a local minimum at $v = 0$, and there exist $\rho \in (0, r)$ and $\alpha(c) > 0$ such that

$$\Phi(c, v) \geq \alpha(c) > 0 \,, \qquad \forall v \in \partial B_\rho^V \,. \tag{5.6}$$

The maximum

$$\overline{m}(c) := \max_{v \in \overline{B_r^V}} \Phi(c, v) \geq \alpha(c) > 0$$

is attained at a point $\overline{v}$ in $B_r^V$ because $\Phi(c, \cdot)$ is negative on $\partial B_r^V$ by (5.5). Furthermore, $\overline{v} \neq 0$ because $\Phi(c, \overline{v}) = \overline{m}(c) > 0$ and $\Phi(c, 0) = 0$. To find another geometrically distinct (cfr. Def. 5.1) non trivial critical point of $\Phi(c, \cdot)$ we define the Mountain Pass critical level

$$\underline{m}(c) := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(c, \gamma(t)) \tag{5.7}$$

where the minimax class $\Gamma$ is

$$\Gamma := \Big\{ \gamma \in C([0,1], \overline{B_r^V}) \ : \ \gamma(0) = 0 \text{ and } \gamma(1) \in \partial B_r^V \Big\} \,. \tag{5.8}$$

Since any path $\gamma \in \Gamma$ intersects $\{v \in V \ : \ \|v\| = \rho\}$, by (5.6),

$$\underline{m}(c) \geq \alpha(c) > 0 \,. \tag{5.9}$$

To prove that $\underline{m}(c)$ is a critical value, we can not directly apply the Mountain Pass Theorem of Ambrosetti-Rabinowitz [1] because $\Phi(c, \cdot)$ is defined only in a neighborhood of 0. However, since

$$\underline{m}(c) \geq \alpha(c) > 0 > -\beta \geq \Phi(c, \cdot)_{|\partial B_r^V} \,, \tag{5.10}$$

we adapt its proof showing that $\underline{m}(c)$ is a critical value. The following lemma holds.

**Lemma 5.2.** *There exists a Palais-Smale sequence $\{v_n\}_{n \geq 0} \subset B_r^V$ at the level $\underline{m}(c)$, i.e. such that*

$$\Phi(c, v_n) \to \underline{m}(c) \,, \quad \nabla_v \Phi(c, v_n) \to 0 \ \textit{ as } n \to \infty \,. \tag{5.11}$$

The proof is based on a classical deformation argument, that we report in detail in Appendix A for completeness (for readers not acquainted with critical point theory).

As a corollary of Lemma 5.2, there exists a non trivial critical point $\underline{v} \in B_r^V$ of $\Phi(c, \cdot)$ at the level $\underline{m}(c)$. Indeed, by compactness, up to subsequence, $v_n$ converges to some $\underline{v}$ belonging to $\overline{B_r^V}$. Actually $\underline{v}$ belongs to $B_r^V \setminus \{0\}$ because, by (5.11), $\Phi(c, \underline{v}) = \underline{m}(c) > 0$ and $\Phi(c, \cdot)_{|\partial B_r^V} < 0$ (by (5.5)) and $\Phi(c, 0) = 0$.

If $\underline{m}(c) < \overline{m}(c)$ then $\Phi(c, \cdot)$ has two geometrically distinct critical points (since, by (4.17), geometrically non distinct critical points have the same value of $\Phi(c, \cdot)$). If $\underline{m}(c) = \overline{m}(c)$ then $\overline{m}(c)$ equals the maximum of $\Phi(c, \cdot)$ over *every* curve in $\Gamma$. Therefore there is a maximum of $\Phi(c, \cdot)$ on each curve $\gamma \in \Gamma$ defined in (5.8) and then there are infinitely many geometrically distinct critical points of $\Phi(c,\cdot)$. Notice that any $O(2) = \mathbb{S}^1 \rtimes \mathbb{Z}_2$ orbit is 1-dimensional and therefore cannot separate the four dimensional domain $B_r^V$ of $\Phi(c,v)$. In any case alternative $(ii)$ holds.

*Case (b).* In this case, since $\Phi(c_*,\cdot)$ takes on both positive and negative values near $v=0$, the functional $\Phi(c,\cdot)$ possesses the Mountain-Pass geometry both for $c > c_*$ and $c < c_*$. The construction is more subtle than the previous case because a level set of $\Phi(c_*,\cdot)$ no longer bounds a compact neighborhood of $v = 0$.

By Proposition A.2 we associate to the degenerate isolated critical point $v = 0$ of $\Phi(c_*, v)$ an arbitrarily small "stable Conley isolating block" $W$ (also called a Gromoll-Meyer set) and its "exit set"

$$W_- := \{v \in W \ : \ \eta^t(v) \notin W \ \forall t > 0 \text{ near } 0\}$$

where $\eta^t(v)$ is the negative gradient flow generated by $-\nabla_v \Phi(c_*, v)$. The exit set $W_-$ lies in a negative level set of $\Phi(c_*,\cdot)$, i.e. $\Phi(c_*,\cdot)_{|W_-} < 0$, and it is not empty since $\Phi(c_*, v)$ assumes negative values arbitrarily close to $v = 0$.

Define the Mountain Pass level

$$m(c) := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(c, \gamma(t))$$

where

$$\Gamma := \Big\{\gamma \in C([0,1], W) \ : \ \gamma(0) = 0 \text{ and } \gamma(1) \in W_-\Big\} \tag{5.12}$$

(note that by Proposition A.2-Item **(II)** there is a continuous path joining 0 and $W_-$).

For $c > c_*$ we deduce that $m(c) \geq \alpha(c) > 0$, arguing as for (5.9). By a deformation argument we deduce the existence of a Palais-Smale sequence (see Appendix A).

**Lemma 5.3.** *There is a Palais-Smale sequence $\{v_n\}_{n\geq 0} \subset W$ at the level $m(c)$, i.e. such that*

$$\Phi(c, v_n) \to m(c)\,, \quad \nabla_v \Phi(c, v_n) \to 0 \ \text{ as } n \to \infty\ .$$

By compactness $v_n$ converges to a critical point $\overline{v} \in W \subset B_r$ of $\Phi(c, v)$. Again $\overline{v} \neq 0$ because $\Phi(c, \overline{v}) = m(c) > 0$ and $\Phi(c, 0) = 0$.

For $c < c_*$ we repeat the previous argument for $-\Phi(c, v)$ (note that $-\Phi(c_*, v)$ assumes negative values arbitrarily close to $v = 0$). The proof of Theorem 1.2 is complete.

**Remark 5.4.** The number of geometrically distinct critical points proved in Theorem 1.2 coincides with the number provided in [16] (which is expected to be the optimal one) in the present case in which $\dim V = 4 = 2n$.

## 5.2 Stokes waves parametrized by the momentum

In this section we prove Theorem 1.3. For definiteness in the sequel we assume that $j_*, j < 0$ (recall that $j, j_*$ have the *same* sign by (2.23)) and $a > 0$. The other case follows similarly.

We define the functional

$$\Psi_a(c, u) := \mathcal{H}(u) + c(\mathcal{I}(u) - a) \tag{5.13}$$

which differs from $\Psi(c, u)$ defined in (1.17) just by a constant. Thus $\mathrm{d}_u\Psi_a(c, u) = \mathrm{d}_u\Psi(c, u)$ and therefore a critical point of $u \mapsto \Psi_a(c, u)$ is a solution of (1.15).

Now we do not fix the speed $c$, as in the previous section, but we look for $c(v)$ such that

$$(\mathrm{d}_v\Phi)(c(v), v)[v] = 0\,, \tag{5.14}$$

for any $v \neq 0$ sufficiently small, namely such that the radial derivative of the function $v \mapsto \Phi(c, v)$ defined in (4.16) vanishes. We mention that the choice for $c(v)$ in [12] is different.

**Lemma 5.5.** *For any $v \in B_r^V \setminus \{0\}$ (with a possibly smaller $r > 0$) there exists a unique $c(v) \in \mathbb{R}$ solving* (5.14) *and satisfying the following properties: the function $v \mapsto c(v)$ is analytic in $B_r^V \setminus \{0\}$,*

$$c(v) = c_* + \mathcal{O}(\|v\|)\,, \quad \mathrm{d}_v c(v) = \mathcal{O}(1) \quad \textit{as } v \to 0\,, \tag{5.15}$$

*and $c(\mathscr{S}v) = c(v)$, $c(\tau_\theta v) = c(v)$ for any $\theta \in \mathbb{R}$.*

*Proof.* By Lemma 4.3 we have that

$$(\mathrm{d}_v\Phi)(c,v)[v] = (c-c_*)\mathrm{d}\mathcal{I}(v)[v] + \mathrm{d}_v G_{\geq 3}(c,v)[v] = 2(c-c_*)\underbrace{\mathcal{I}(v)}_{=\|v\|_*^2} + \mathtt{G}_{\geq 3}(c,v) \tag{5.16}$$

where $\mathtt{G}_{\geq 3}(c,v)$ is an analytic function in $B_r(c_*) \times B_r^V$ satisfying

$$\mathrm{d}^\ell \mathtt{G}_{\geq 3}(c,0) = 0\,, \quad \forall \ell = 0,1,2,\,, \quad \forall c \in (c_* - r, c_* + r)\,. \tag{5.17}$$

In view of (5.16) and setting $c = c_* + \Delta$, the equation (5.14) is equivalent, for any $v \neq 0$, to look for a fixed point of

$$\Delta = -\frac{\mathtt{G}_{\geq 3}(c_* + \Delta, v)}{2\|v\|_*^2} =: F(\Delta, v)\,.$$

By (5.17) there exists $K > 0$ such that for any $\Delta \in B_r(0)$, any $v \in B_r^V \setminus \{0\}$,

$$|F(\Delta,v)| \leq K\|v\|\,, \quad |\partial_\Delta F(\Delta,v)| = \frac{|\partial_c \mathtt{G}_{\geq 3}(c_* + \Delta, v)|}{2\|v\|_*^2} \leq K\|v\|\,.$$

As a consequence for any $0 < \|v\| < r/K < 1$, the map $F(\cdot, v)$ is a contraction on $B_r(0)$. Hence for any $v \in B_{r/K}^V(0) \setminus \{0\}$ there exists a unique fixed point $\Delta(v)$ of $F(\cdot, v)$, namely a solution $c(v) := c_* + \Delta(v)$ of (5.14) (in the smaller domain $r/K$).

The function $v \mapsto c(v) = c_* + \Delta(v)$ is analytic by applying the implicit function theorem to the analytic function $H(\Delta, v) := \Delta - F(\Delta, v)$ which vanishes at $H(\Delta(v), v) = 0$ and satisfies $\partial_\Delta H(\Delta, v) = 1 - \partial_\Delta F(\Delta, v) \neq 0$.

The function $\Delta(v) := c(v) - c_*$ satisfies $|\Delta(v)| = |F(\Delta(v), v)| \leq K\|v\|$ proving the first bound in (5.15). Taking the differential of the equation (5.14) which is satisfied identically for $c = c(v)$ and using (5.16)

$$\mathrm{d}_v c(v)\big(2\|v\|_*^2 + \partial_c G_{\geq 3}(c(v), v)\big) + 2\big(c(v) - v\big)\mathrm{d}_v \mathcal{I}(v)[\widehat{v}] + \mathrm{d}_v G_{\geq 3}(c(v), v)[\widehat{v}] = 0\,.$$

Thus, for $v$ sufficiently small, using $c(v) - c_* = \mathcal{O}(\|v\|)$ we obtain $\|\mathrm{d}_v c(v)\| = \mathcal{O}(1)$ as $v \to 0$. This proves the second bound in (5.15). The last invariance property follows by (4.17) and uniqueness. □

Next we define the set

$$\mathcal{S}_a := \mathcal{S}_{r,a} := \Big\{ v \in B_r^V \,:\, I(v) := \mathcal{I}(v + w(c(v), v)) = a \Big\}\,. \tag{5.18}$$

Since $I(v)$ is asymptotic, for $v \to 0$, to the homogeneous quadratic function $\mathcal{I}(v)$ in (5.4), the set $\mathcal{S}_a$ is, for $a > 0$ small, an ellipsoid-like compact manifold. A supplementary to the tangent space $T_v\mathcal{S}_a$ is the 1 dimensional space spanned by $\langle v \rangle$, namely $V = T_v\mathcal{S}_a \oplus \langle v \rangle$.

**Lemma 5.6.** *There exist $r_0, a_0 > 0$ such that for any $r \in (0, r_0)$ and $a \in (0, a_0)$, the set $\mathcal{S}_a$ in* (5.18) *is a compact manifold contained in $\mathcal{A}_a := \{v \in B_r^V \,:\, (a/2)^{1/2} \leq \|v\|_* \leq (2a)^{1/2}\}$.*

*Proof.* By (4.8), (5.15), (5.4), the functional $I(v)$ in (5.18) has the expansion

$$I(v) = \mathcal{I}(v) + \mathcal{O}(\|v\|^3) = \|v\|_*^2 + \mathcal{O}(\|v\|^3) \tag{5.19}$$

for any $\|v\| \leq r < r_0$ small enough. As a consequence there is $a_0 > 0$ such that, for any $0 < a < a_0$, any $v$ in $\mathcal{S}_a$ satisfies $\sqrt{a/2} \leq \|v\|_* \leq \sqrt{2a}$. The set $\mathcal{S}_a$ is thus contained in the annulus $\mathcal{A}_a$ and closed (the function $I(v)$ is continuous), thus compact. On the set $\mathcal{A}_a$ the function $c(v)$ is actually analytic. We now prove that $\mathcal{S}_a$ is a manifold. Differentiating $I(v)$ in (5.18) at any $v \in B_r^V$ in the direction $v$ (radial derivative), we have

$$\mathrm{d}I(v)[v] = \mathrm{d}\mathcal{I}(v + w(c(v), v))\big[v + \mathrm{d}_v w(c(v), v)[v] + \partial_c w(c(v), v)\mathrm{d}_v c(v)[v]\big]\,.$$

By (4.8), (5.15), (5.4), there exist constants $0 < c_1 < c_2$ such that

$$c_1\|v\|_*^2 \leq \mathrm{d}_v I(v)[v] \leq c_2\|v\|_*^2 \tag{5.20}$$

for any $\|v\|_* \leq r < r_0$. As a consequence the set $\mathcal{S}_a$ is a manifold. □

Finally, for any $a \in (0, a_0)$ defined in Lemma 5.6, we define the functional

$$\phi_a : B_r^V \to \mathbb{R}\,, \quad \phi_a(v) := \Psi_a\big(c(v), v + \breve{w}(v)\big) \tag{5.21}$$

where $\breve{w}(v) := w(c(v), v)$. Then $\phi_a(\tau_\theta v) = \phi_a(v)$ for any $\theta$ and $\phi_a(\mathscr{S}v) = \phi_a(v)$.

**Lemma 5.7.** *If $\overline{v} \in \mathcal{S}_a$ is a critical point of $\phi_a : \mathcal{S}_a \to \mathbb{R}$ then $\overline{u} := \overline{v} + \breve{w}(\overline{v})$ is a solution of $\mathcal{F}(c(\overline{v}), u) = 0$ with momentum $\mathcal{I}(\overline{u}) = a$ and speed $c(\overline{v})$.*

*Proof.* Differentiating (5.21) at any $v \in \mathcal{S}_a$, in any direction $\widehat{v} \in V$,

$$\begin{aligned}
\mathrm{d}_v\phi_a(v)[\widehat{v}] &= (\partial_c\Psi_a)(c(v), \breve{w}(v))\,\mathrm{d}_v c(v)[\widehat{v}] + \mathrm{d}_u\Psi(c(v), v + \breve{w}(v))[\widehat{v}] \\
&\qquad + \underbrace{\mathrm{d}_u\Psi(c(v), v + \breve{w}(v))[\mathrm{d}_v\breve{w}(v)[\widehat{v}]]}_{=0 \text{ by (4.19) and } \mathrm{d}_v\breve{w}(v)[\widehat{v}]\in W} \\
&\overset{(5.13)}{=} \underbrace{(\mathcal{I}(v + \breve{w}(v)) - a)}_{=0 \text{ by (5.18)}}\, \mathrm{d}_c c(v)[\widehat{v}] + \mathrm{d}_u\Psi(c(v), v + w(c(v), v))[\widehat{v}] \\
&\overset{(4.18)}{=} (\mathrm{d}_v\Phi)(c(v), v)[\widehat{v}]\,.
\end{aligned} \tag{5.22}$$

Let $\overline{v}$ be a critical point of $\phi_a : \mathcal{S}_a \to \mathbb{R}$. Then, by (5.22) and recalling (5.18), there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$\mathrm{d}_v\Phi(c(\overline{v}), \overline{v})[\widehat{v}] = \mu\, \mathrm{d}_v I(\overline{v})[\widehat{v}]\,, \quad \forall \widehat{v} \in V\,. \tag{5.23}$$

Taking $\widehat{v} = \overline{v}$ inside (5.23) we get

$$0 \overset{(5.14)}{=} \mathrm{d}_v\Phi(c(\overline{v}), \overline{v})[\overline{v}] = \mu \mathrm{d}_v I(\overline{v})[\overline{v}]$$

and, since $\mathrm{d}_v I(v)[v] \neq 0$ by (5.20) and $\|v\|_*^2 \geq a/2 > 0$, we deduce that $\mu = 0$. Therefore, by (5.23), $\overline{v}$ is a critical point of $v \mapsto \Phi(c(\overline{v}), v)$ and Lemma 4.2 implies Lemma 5.7. □

Since $\mathcal{S}_a$ is a compact manifold (Lemma 5.6), the functional $\phi_a$ on $\mathcal{S}_a$ possesses at least a minimum $m$ and maximum $M$. If $m < M$ a minimum point $\underline{v}$, i.e. $\phi_a(\underline{v}) = m$, and a maximum point $\overline{v}$, i.e. $\phi_a(\overline{v}) = M$, are geometrically distinct. If $m = M$ the function $\phi_a$ is constant on $\mathcal{S}_a$ and then there are infinitely many geometrically distinct critical points of $\phi_a$ (any $O(2) = \mathbb{S}^1 \rtimes \mathbb{Z}_2$ orbit is 1-dimensional and $\mathcal{S}_a$ is 3-dimensional). In both cases, in view of Lemma 5.7, Theorem 1.3 is proved.

**Remark 5.8.** The number of 2 geometrically distinct critical points obtained in Theorem 1.3 by topological arguments (the maximum and the minimum) is in general optimal. The function

$$f : \mathbb{S}^3 := \{(z_1, z_2) \in \mathbb{C}^2 \ : \ |z_1|^2 + |z_2|^2 = 1\} \to \mathbb{R}\,, \ (z_1, z_2) \mapsto |z_1|^2\,,$$

is invariant under the actions of the group $\mathbb{S}^1 \rtimes \mathbb{Z}_2$ where $\tau_\theta(z_1, z_2) := (e^{-\mathrm{i}j_1\theta} z_1, e^{-\mathrm{i}j_2\theta} z_2)$ and $\mathscr{S}(z_1, z_2) := (\overline{z}_1, \overline{z}_2)$, cfr. (2.19). The function $f$ has only two critical orbits $\{(z_1, 0) \ : \ |z_1| = 1\}$ and $\{(0, z_2) \ : \ |z_2| = 1\}$.

# A Appendix

In this appendix we prove Lemmata 5.2 and 5.3 and the existence of a stable Conley isolating block.

## A.1 Existence of Palais-Smale sequences

The argument is based on deforming the sublevels of $\Phi(c, \cdot)$ and exploits a topological change between $\{\Phi(c, v) \le \underline{m}(c) + \mu\}$ and $\{\Phi(c, v) \le \underline{m}(c) - \mu\}$: in view of (5.7) the first set is path connected whereas the second one is not. For simplicity of notation we denote $\nabla\Phi = \nabla_v\Phi$.

Proof of Lemma 5.2. We claim the following.
**Claim:** *For any $0 < \mu < \underline{m}(c)/2$ there exists $v \in B_r^V$ such that*

$$\underline{m}(c) - \mu \le \Phi(c, v) \le \underline{m}(c) + \mu \quad \textit{and} \quad \|\nabla\Phi(c, v)\| < 2\mu\,. \tag{A.1}$$

Then, choosing $\mu = 1/n$ for any $n$ large enough we find a Palais-Smale sequence $v_n$ at the level $\underline{m}(c)$, i.e. satisfying (5.11).

**The Deformation Argument.** There is $\delta > 0$ such that the functional $\Phi(c, v)$ is defined on the open ball $B_{r+\delta}^V$ and it is negative on the annulus $B_{r+\delta}^V \setminus B_{r-\delta}^V$ by (5.5).

For any $0 < \mu < \underline{m}(c)/2$, we define the sets

$$\tilde{N} := \{v \in B_{r+\delta}^V \ : \ |\Phi(c, v) - \underline{m}(c)| \le \mu \ \text{ and } \ \|\nabla\Phi(c, v)\| \ge 2\mu\} \tag{A.2}$$

$$N := \{v \in B_{r+\delta}^V \ : \ |\Phi(c, v) - \underline{m}(c)| < 2\mu \ \text{ and } \ \|\nabla\Phi(c, v)\| > \mu\}\,. \tag{A.3}$$

Clearly $\tilde{N} \subset N$. Note also that $\partial B_r^V \subset N^c := B_{r+\delta}^V \setminus N$ since, for any $v \in \partial B_r^V$ we have $\Phi(c, v) < 0$ by (5.5), $0 < 2\mu < \underline{m}(c)$, and then $v \in N^c$.

The sets $\tilde{N}$ and $N^c$ are closed and disjoints and therefore there is a locally Lipschitz non-negative function $g : B_{r+\delta}^V \to [0, 1]$ such that

$$g = 1 \text{ on } \tilde{N}\,, \qquad g = 0 \text{ on } N^c\,, \tag{A.4}$$

for example $g(v) := \frac{d(v,N^c)}{d(v,N^c)+d(v,\tilde{N})}$ where $d(\cdot, \cdot)$ denotes the distance function in $V$.

We define the locally Lipschitz and bounded vector field

$$X(c, v) := -g(v)\frac{\nabla\Phi(c, v)}{\|\nabla\Phi(c, v)\|} \tag{A.5}$$

which is well defined because if $\|\nabla\Phi(c, v)\| < \mu$ then $v \in N^c$ (cfr. (A.3)) and so $g(v) = 0$ by (A.4). Furthermore the vector field $X(c, \cdot)$ vanishes on $\partial B_r^V$ since $\partial B_r^V \subset N^c$.

For each $v \in B_{r+\delta}^V$, the unique solution of the Cauchy problem

$$\frac{d}{dt}\eta^t(v) = X(c, \eta^t(v))\,, \quad \eta^0(v) = v\,, \tag{A.6}$$

is defined for any $t \in \mathbb{R}$ (since $X$ is bounded) and (*i*) $\eta^t(\cdot)$ is a homeomorphism of $B^V_{r+\delta}$.

Furthermore, since $X(c,v) = 0$ for any $v \in N^c$, by (A.5) , (A.4),

(*ii*) $\eta^t(v) = v$, for any $t$, if $|\Phi(c,v) - \underline{m}(c)| \geq 2\mu$ or if $\|\nabla\Phi(c,v)\| \leq \mu$, in particular $\eta^t(0) = 0$ and $\eta^t(v) = v$ for any $v \in \partial B^V_r \subset N^c$.

The properties (*i*)-(*ii*) imply that the min-max class $\Gamma$ defined in (5.8) is invariant under the flow of $X(c,\cdot)$, namely

$$\text{for any path } \gamma \in \Gamma, \text{ for any } t \in \mathbb{R}, \text{ the deformed path } \eta^t \circ \gamma \text{ belongs to } \Gamma. \tag{A.7}$$

Furthermore, by (A.5), (A.6),

(*iii*) for any $v \in B^V_{r+\delta}$, for any $t \in \mathbb{R}$

$$\frac{d}{dt}\Phi(c,\eta^t(v)) = -g(\eta^t(v))\|\nabla\Phi(c,\eta^t(v))\| \leq 0\,. \tag{A.8}$$

We now prove the claim (A.1). Arguing by contradiction suppose there exists $0 < \mu < \underline{m}(c)/2$ such that

$$\{v \in B^V_r \ : \ \underline{m}(c) - \mu \leq \Phi(c,v) \leq \underline{m}(c) + \mu\} \subseteq \{v \in B^V_r \ : \ \|\nabla\Phi(c,v)\| \geq 2\mu\}\,. \tag{A.9}$$

By the definition of $\underline{m}(c) > 0$ in (5.7) there exists a path $\gamma \in \Gamma$ (see (5.8)) such that

$$\max_{t\in[0,1]} \Phi(c,\gamma(t)) \leq \underline{m}(c) + \mu\,. \tag{A.10}$$

But we claim that

$$\Phi(c,\eta^1(\gamma([0,1])) \leq \underline{m}(c) - \mu\,, \tag{A.11}$$

implying the contradiction

$$\underline{m}(c) := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \Phi(c,\gamma(t)) \leq \max_{t\in[0,1]} \Phi(c, \underbrace{\eta^1(\gamma(t))}_{\in\Gamma \text{ by (A.7)}}) \leq \underline{m}(c) - \mu\,.$$

Let us prove (A.11). Pick a point $v \in \gamma([0,1])$. If $\Phi(c,\eta^t(v)) \leq \underline{m}(c) - \mu$ for some $0 \leq t \leq 1$ thus $\Phi(c,\eta^1(v)) \leq \underline{m}(c) - \mu$ since $\Phi(c,\eta^t(v))$ is not-increasing by (A.8). If $\Phi(c,\eta^t(v)) > \underline{m}(c) - \mu$ for any $t \in [0,1]$, then, by (A.9), (A.10), (A.8),

$$\underline{m}(c) - \mu \leq \Phi(c,\eta^t(v)) \leq \underline{m}(c) + \mu \quad \text{and} \quad \|\nabla\Phi(c,\eta^t(v))\| \geq 2\mu\,, \tag{A.12}$$

i.e. $\eta^t(v) \in \tilde{N}$ in (A.2), for any $t \in [0,1]$. By (A.8), since $g \equiv 1$ on $\tilde{N}$ (cfr. (A.4))

$$\begin{aligned}\Phi(c,\eta^1(v)) &= \Phi(c,v) - \int_0^1 g(\eta^t(v))\,\|\nabla\Phi(c,\eta^t(v))\|dt \\ &\overset{(A.10)}{\leq} (\underline{m}(c) + \mu) - \int_0^1 \|\nabla\Phi(c,\eta^t(v))\|dt \overset{(A.12)}{\leq} \underline{m}(c) + \mu - 2\mu = \underline{m}(c) - \mu\end{aligned}$$

proving (A.11). This contradiction concludes the proof of Lemma 5.2.

Proof of Lemma 5.3. To apply the same argument of Lemma 5.2 the main issue is to prove that the min-max class $\Gamma$ defined in (5.12) is invariant under the positive flow $\eta^t(v)$ generated by the vector field $X(c,v) := -g(v)\nabla\Phi(c,v)$ where $g(v)$ is the scalar non-negative function defined as in (A.4) (clearly defining $N, \tilde N$ in (A.2), (A.3) as subsets of $W$). This is true, for $c$ sufficiently close to $c_*$ by the last statement of Proposition A.2. If $v \in W_-$ then $\Phi(c_*, v) < 0$ and, for $c$ sufficiently close to $c_*$, also $\Phi(c,v) < 0$ and thus $X(c,v) \equiv 0$ (because $g(v) = 0$ on $N^c$). Therefore $\eta^t(v) = v$ for any $t$ and $\eta^t(\cdot)_{|W_-} = I$. On the other hand, the set $W \setminus W_-$ is invariant for the positive flow $\eta^t(v)$ generated by $X(c,v)$ and consequently any curve $\gamma(\cdot)$ with values in $W$ is deformed by $\eta^t(\cdot)$ into a curve with values in $W$, proving the invariance of the min-max class $\Gamma$ in (5.12) under the deformations $\eta^t(\cdot)$.

## A.2 A stable Conley isolating block

Let $f : B_r \subset V \to \mathbb{R}$ be a function of class $C^2$ defined in the ball $B_r$ of radius $r > 0$ centered at 0 in a finite dimensional Hilbert space $V$ with an isolated critical point at $v = 0$ at the level $f(0) = 0$. We follow the construction in [7], [19]. Let $\varepsilon > 0$ and $\delta \in (0, r)$ such that 0 is the unique critical value in $[-\varepsilon, \varepsilon]$ and $v = 0$ the unique critical point of $f$ in $\overline{B_\delta}$. Consider the function

$$g(v) := \lambda\|v\|^2 + f(v) \tag{A.13}$$

where

$$0 < \lambda < \frac{\beta}{2\delta}, \qquad \beta := \min_{\frac{\delta}{2} \le \|v\| \le \delta} \|\nabla f(v)\| > 0\,. \tag{A.14}$$

We denote $g_\mu := \{v \in B_r\ : g(v) \le \mu\}$. Given $\gamma \in (0, \varepsilon)$, $\mu > 0$ we define the sets

$$W := f^{-1}[-\gamma, \gamma] \cap g_\mu = \big\{v \in B_r\ :\ |f(v)| \le \gamma,\ g(v) \le \mu\big\}, \tag{A.15}$$
$$W_- := f^{-1}(-\gamma) \cap W = \big\{v \in B_r\ :\ f(v) = -\gamma,\ g(v) \le \mu\big\}\,. \tag{A.16}$$

**Lemma A.1.** *If $\gamma, \mu$ satisfy*

$$0 < \gamma < \min\Big\{\varepsilon, \frac{3\delta^2\lambda}{8}\Big\}, \quad \frac{\delta^2\lambda}{4} + \gamma < \mu < \delta^2\lambda - \gamma\,, \tag{A.17}$$

*then*

*(i)* $\overline{B_{\delta/2}} \cap f^{-1}[-\gamma,\gamma] \subset W \subset B_\delta \cap f^{-1}[-\varepsilon, \varepsilon]$;

*(ii)* $f^{-1}[-\gamma,\gamma] \cap g^{-1}(\mu) \subset B_\delta \setminus \overline{B_{\delta/2}}$;

*(iii)* $\langle \nabla g(v), \nabla f(v)\rangle > \beta(\beta - 2\delta\lambda) > 0$ *for any* $v \in \overline{B_\delta} \setminus B_{\delta/2}$.

*Proof.* Properties $(i)$-$(ii)$ are directly verified by the definitions (A.15), (A.16) using (A.17) and noting that $\|v\|^2 = \frac{\mu - f(v)}{\lambda}$ for any $v \in g^{-1}(\mu)$. Property $(iii)$ follows by (A.13) and (A.14). □

The set $W$ is a compact neighborhood of $v = 0$. The set $W_-$ is a closed subset of $W$.

**Proposition A.2. (Stable Conley isolating block)**. *The set $W$ in* (A.15) *has the following properties:*

- **(I)** *for any $v \in W$ then either*
  - *(i)* $f(v) = -\gamma$, or
  - *(ii)* $\eta^t(v) \in W$*, for any $t > 0$ near 0.*

- **(II)** *The set* $W_-$ *in* (A.16) *is the "exit" set of* $W$ *with respect to the negative gradient flow* $\eta^t(v)$ *generated by* $-\nabla f(v)$*, namely*

$$W_- = \big\{ v \in W \ : \ \eta^t(v) \notin W \ \forall t > 0 \ \text{near } 0 \big\} . \tag{A.18}$$

  *If* $f$ *assumes negative values arbitrarily close to* $v = 0$ *then* $W_- \neq \emptyset$ *and there is a point* $v \in W_-$ *such that* $\eta^t(v) \to 0$ *as* $t \to -\infty$*.*

- **(III)** *If* $F : B_r \to \mathbb{R}$ *is sufficiently close in* $C^1$ *norm to* $f$ *on* $\partial W \setminus W_-$ *then, for any* $v \in W \setminus W_-$ *the negative gradient flow* $\eta_F^t$ *generated by* $-\nabla F(v)$ *satisfies* $\eta_F^t(v) \in W$ *for any* $t > 0$ *close to* $0$*.*

*Proof.* Let us prove item **(I)**. Let $v \in W$. Either $f(v) = -\gamma$ (case $(i)$) or $f(v) \in (-\gamma, \gamma]$ case $(ii)$. In this case $\gamma \geq f(v) \geq f(\eta^t(v)) > -\gamma$ for any $t > 0$ near 0. Furthermore, if $g(v) < \mu$ then by continuity $g(\eta^t(v)) < \mu$ for any $t > 0$ near 0. If $g(v) = \mu$ then by Lemma A.1-$(ii)$ we deduce that $v \in \overline{B_\delta} \setminus B_{\delta/2}$ and thus Lemma A.1-$(iii)$ implies that $\frac{d}{dt} g(\eta^t(v))_{|t=0} = -\langle \nabla g(v), \nabla f(v) \rangle < 0$. In both cases $g(\eta^t(v)) < \mu$ for any $t > 0$ near 0, thus $\eta^t(v) \in W$ for any $t > 0$ near 0.

The equality in (A.18) follows by Item **(I)** and since for any $v \in W_-$ we have $\nabla f(v) \neq 0$ we deduce (A.18). The last statement of Item **(II)** follows because the $\omega$ and $\alpha$ limit sets of the gradient flow is not empty and contained in the set of critical points of $f$, jointly with item **(I)**.

We now decompose $W$ in (A.15) in disjoint subsets as

$$W = \underbrace{\{ v \in B_r \ : \ |f(v)| < \gamma , \ |g(v)| < \mu \}}_{=:A} \cup W_- \cup \mathcal{W}$$

where $W_-$ is defined in (A.16) and

$$\mathcal{W} := \underbrace{\Big\{ v \in B_r \ : \ -\gamma < f(v) \leq \gamma , \ g(v) = \mu \Big\}}_{=:\mathcal{W}_1} \bigcup \underbrace{\Big\{ v \in B_r \ : \ f(v) = \gamma , \ g(v) < \mu \Big\}}_{=:\mathcal{W}_2} . \tag{A.19}$$

Note that the set $\mathcal{W}_1$ in (A.19) is included in $B_\delta \setminus \overline{B_{\delta/2}}$. It results that

$$A = \overset{\circ}{W} \qquad \text{and} \qquad W_- \cup \mathcal{W} = \partial W . \tag{A.20}$$

Indeed $A \subset \overset{\circ}{W}$ trivially by continuity and we claim that any $v \in W_- \cup \mathcal{W}$ belongs to the boundary of $W$. If $v \in W_-$ the flow $\eta^t(v) \notin W$ for any $t > 0$ arbitrarily small. If $v \in \mathcal{W}_1$ Lemma A.1-$(ii)$-$(iii)$ implies that $\frac{d}{dt} g(\eta^t(v))_{|t=0} = -\langle \nabla g(v), \nabla f(v) \rangle < 0$, so $g(\eta^t(v)) > \mu$ for any $t < 0$ arbitrarily small. If $v \in \mathcal{W}_2$ then $\frac{d}{dt} f(\eta^t(v))_{|t=0} = -\|\nabla f(v)\|^2 < 0$ so $f(\eta^t(v)) > \gamma$ for any $t < 0$ arbitrarily small. This implies (A.20).

Let us finally prove item **(III)**. Using that $f$ has no critical points on the part of the boundary $\mathcal{W}$ and that $\langle \nabla g(v), \nabla f(v) \rangle > \beta(\beta - 2\delta\lambda) > 0$ for any $v \in \mathcal{W}_1 \subset \overline{B_\delta} \setminus B_{\delta/2}$ by Lemma A.1, we deduce that, for $F$ sufficiently close to $f$,

$$\langle \nabla f(v), \nabla F(v) \rangle > 0 , \quad \forall v \in \mathcal{W} , \quad \langle \nabla g(v), \nabla F(v) \rangle > 0 , \quad \forall v \in \mathcal{W}_1 . \tag{A.21}$$

By (A.21) for any $v \in \mathcal{W}_1 \cup \mathcal{W}_2$ we have that $\eta_F^t(v) \in \overset{\circ}{W}$ for any $t > 0$ close to 0. Thus the flow $\eta_F^t(v)$ can exit $W$ only through $W_-$ and the proposition is proved. $\square$

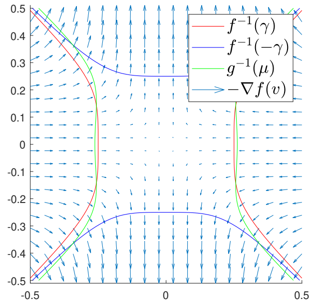


Figure 2: A stable Conley isolating block for the function $f(x, y) = x^4 - y^4$. For any function $F$ which is $C^1$-close to $f$, the negative gradient vector field $-\nabla F$ points inward $W$ on $\partial W \setminus W_-$ and points outside $W$ on $W_-$.

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