THE MONOTONICITY OF THE SYSTOLE OF CONVEX RIEMANNIAN TWO-SPHERES

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Abstract. We prove that the systole of the smooth boundary of a strictly convex ball in $\mathbb{R}^3$ is monotone with respect to the inclusion.

Throughout this note, the notion of convexity must be understood in the differentiable sense: A compact three-ball $B \subset \mathbb{R}^3$ with smooth boundary is strictly convex when there exists a smooth function $F : \mathbb{R}^3 \to [0, \infty)$ with positive definite Hessian at every point and such that $\partial B = F^{-1}(1)$. Equivalently, the boundary sphere $M = \partial B$, which will always be equipped with the Riemannian metric g that is the restriction of the ambient Euclidean metric, has strictly positive Gaussian curvature. The systole $\mathrm{sys}(M) > 0$ is the length of the shortest closed geodesic of (M, g). Our main result answers in dimension 3 a question that was posed to us by Yaron Ostrover:

Theorem 0.1. *Let $B_1 \subseteq B_2$ be two compact strictly convex three-balls in $\mathbb{R}^3$ with smooth boundary. Then* $\mathrm{sys}(\partial B_1) \leq \mathrm{sys}(\partial B_2)$.

The main ingredient of the proof is the observation that the systole of positively curved Riemannian two-spheres coincides with the classical Birkhoff min-max, as we will now prove. Let (M, g) be a Riemannian two-sphere. We denote the energy functional on the $W^{1,2}$ free loop space by

$$E : \Lambda M = W^{1,2}(S^1, M) \to [0, \infty), \qquad E(\zeta) = \int_{S^1} \|\dot\zeta(t)\|_g^2 \mathrm{d}t.$$

Here and in the following, we denote by $S^1 = \mathbb{R}/\mathbb{Z}$ the 1-periodic circle. We consider the unit sphere $S^2 \subset \mathbb{R}^3$. For each $z \in [-1, 1]$, we denote by $\gamma_z : S^1 \to S^2$ the parallel at latitude z, parametrized as

$$\gamma_z(t) = \left(\sqrt{1 - z^2} \cos(2\pi t), \sqrt{1 - z^2} \sin(2\pi t), z\right).$$

For each continuous map $u : [-1, 1] \to \Lambda M$ such that $E(u(0)) = E(u(1)) = 0$ there exists a unique continuous map $\tilde u : S^2 \to M$ such that $u(z) = \tilde u \circ \gamma_z$ for each $z \in [-1, 1]$. We denote by $\mathcal{U}$ the space of such maps u whose associated $\tilde u$ has degree 1. The Birkhoff min-max value

$$\mathrm{bir}(M, g) = \inf_{u \in \mathcal{U}} \max_{z \in [-1,1]} E(u(z))^{1/2}$$

is the length of some closed geodesic of (M, g).

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Lemma 0.2. *On every positively curved closed Riemannian two-sphere* (M, g)*, we have* $\mathrm{bir}(M, g) = \mathrm{sys}(M, g)$.

Proof. Let $\gamma : S^1 \to M$ be a shortest closed geodesic of (M, g) parametrized with constant speed, so that $E(\gamma) = L(\gamma)^2 = \mathrm{sys}(M, g)^2$. A theorem of Calabi–Cao [CC92] implies that γ is simple, that is, an embedding $\gamma : S^1 \hookrightarrow M$. We fix an orientation on M, and consider the corresponding complex structure of (M, g). Namely, for every non-zero $v \in T_xM$, the tangent vector $Jv \in T_xM$ is obtained by rotating v in the positive direction of an angle $\pi/2$. We consider the vector field $\nu(t) = J\dot\gamma(t)$ orthogonal to $\dot\gamma(t)$. Notice that ν is a parallel vector field, since the complex structure J is parallel. If K_g denotes the Gaussian curvature of (M, g), we have

$$\mathrm{d}^2E(\gamma)[\nu,\nu] = \int_{S^1} \big(\|\nabla_t\nu\|_g^2 - K_g\|\dot\gamma\|_g^2\|\nu\|_g^2\big)\,\mathrm{d}t = -\int_{S^1} K_g\|\dot\gamma\|_g^4\mathrm{d}t < 0. \tag{0.1}$$

We now consider Morse's finite dimensional approximation of the free loop space (see, e.g., [Mil63]). We fix a positive integer k that is large enough so that $d(\zeta(t_0), \zeta(t_1)) < \mathrm{injrad}(M, g)$ for all $\zeta \in \Lambda M$ with $E(\zeta) \le E(\gamma) = \mathrm{sys}(M, g)^2$ and for all $t_0, t_1 \in \mathbb{R}$ with $|t_1 - t_0| < 1/k$. Here, d denotes the Riemannian distance on (M, g). We consider the open finite dimensional manifold

$$\Lambda_kM = \big\{\boldsymbol{x} = (x_0, ..., x_{k-1}) \in M \times ... \times M \ \big|\ d(x_i, x_{i+1}) < \mathrm{injrad}(M, g) \quad \forall i \in \mathbb{Z}_k\big\}.$$

Such a manifold admits an embedding

$$\iota : \Lambda_kM \hookrightarrow \Lambda M, \qquad \iota(\boldsymbol{x}) = \gamma_{\boldsymbol{x}},$$

where each restriction $\gamma_{\boldsymbol{x}}|_{[i/k,(i+1)/k]}$ is the shortest geodesic parametrized with constant speed joining x_i and x_{i+1}. We denote the restricted energy functional by

$$E_k = E \circ \iota : \Lambda_kM \to [0, \infty), \qquad E_k(\boldsymbol{x}) = k \sum_{i\in\mathbb{Z}_k} d(x_i, x_{i+1})^2.$$

Let $\boldsymbol{x} := \iota^{-1}(\gamma)$. We consider the tangent vector $\boldsymbol{v} := (v_0, ..., v_{k-1}) \in T_{\boldsymbol{x}}(\Lambda_kM)$ such that $v_i = \nu(i/k)$ for all $i \in \mathbb{Z}_k$. Inequality (0.1) readily implies that $\mathrm{d}\iota(\boldsymbol{x})\boldsymbol{v}$ lies in the negative cone of the Hessian $\mathrm{d}^2E(\gamma)$, since

$$\begin{aligned} \mathrm{d}^2E_k(\boldsymbol{x})[\boldsymbol{v},\boldsymbol{v}] &= \tfrac{\mathrm{d}^2}{\mathrm{d}z^2}\big|_{z=0}E(\iota(\exp_{\boldsymbol{x}}(z\boldsymbol{v}))) \\ &\le \tfrac{\mathrm{d}^2}{\mathrm{d}z^2}\big|_{z=0}E(\exp_{\gamma(\cdot)}(z\nu(\cdot))) \\ &= \mathrm{d}^2E(\gamma)[\nu,\nu] \\ &< 0. \end{aligned} \tag{0.2}$$

Here, the exponential map in Λ_kM is the one associated with the natural Riemannian metric $g \oplus ... \oplus g$.

The complement $M \setminus \gamma$ has two connected components B_+ and B_-, each one diffeomorphic to a two-ball. The vector field ν points into one of them, say B_+. We define the continuous map

$$w : [-1/3, 1/3] \to \Lambda_kM, \qquad w(z) = \exp_{\boldsymbol{x}}(z\epsilon\boldsymbol{v}).$$

Notice that $w(0) = \boldsymbol{x}$. We fix $\epsilon > 0$ small enough so that, for all $z \in (0, 1/3]$, the loop $\iota(w(\pm z))$ is entirely contained in the open ball $B_\pm$, and by Equation (0.2) we have

$$E_k(w(z)) < E_k(w(0)) = \mathrm{sys}(M, g)^2, \qquad \forall z \in [-1/3, 1/3] \setminus \{0\}.$$

We now consider the open subspaces $U_+, U_- \subset \Lambda_k M$ given by

$$U_\pm = \Lambda_k M \cap (B_\pm \times ... \times B_\pm).$$

We have $w(\pm 1/3) \in U_\pm$. The flow ϕ_s of the anti-gradient $-\nabla E_k$ is complete in positive time s in the sublevel set $E_k^{-1}([0, \mathrm{sys}(M,g)^2])$. We claim that

$$\phi_s(w(\pm 1/3)) \in U_\pm, \qquad \forall s \geq 0.$$

Indeed, assume by contradiction that there exists $s_0 > 0$ such that $\phi_{s_0}(w(\pm 1/3)) \in \partial U_\pm$, and take s_0 to be the minimal such time. If $\boldsymbol{y} := \phi_{s_0}(w(\pm 1/3))$, the components of the anti-gradient vector $\boldsymbol{z} := -\nabla E_k(\boldsymbol{y})$ are given by

$$z_i = 2(\dot\gamma_{\boldsymbol{y}}(\tfrac{i}{k}^+) - \dot\gamma_{\boldsymbol{y}}(\tfrac{i}{k}^-)), \qquad \forall i \in \mathbb{Z}_k.$$

Since $\boldsymbol{y} \in \partial U_\pm$, at least one of its components y_i must belong to $\partial B_\pm$. Assume that all the y_i's belong to $\partial B_\pm$, and therefore they are of the form $y_i = \gamma(t_i)$ for some $t_i \in S^1$. In this case, we have $z_i = \lambda_i \dot\gamma(t_i)$ for some $\lambda_i \in \mathbb{R}$; but this is impossible, since it would imply that all the components of $\phi_s(w(\pm 1/3))$ belong to $\partial B_\pm$ for all $s \in \mathbb{R}$, and thus that $\phi_s(w(\pm 1/3))$ belong to $\partial U_\pm$ for all $s \in \mathbb{R}$. Therefore at least one component $y_i \in \partial B_\pm$ is adjacent to a component in the interior $y_{i-1} \in B_\pm$. However, this implies that the vector z_i points inside $B_\pm$, and therefore $\phi_{s_0-\delta}(w(\pm 1/3)) \notin U_\pm$ for all $\delta > 0$ small enough, contradicting the minimality of s_0.

We set $\delta := \min\{\mathrm{injrad}(M,g), \mathrm{sys}(M)/(4k)\}$. Since $E_k(\phi_s(w(\pm 1/3))) < \mathrm{sys}(M,g)^2$ for all $s \geq 0$, and since $\mathrm{sys}(M,g)^2$ is the smallest positive critical value of E_k, we can fix a large enough $s > 0$ such that $E_k(\phi_s(w(\pm 1/3))) < \delta^2$. We extend w to a map $w : [-2/3, 2/3] \to \Lambda_k M$ by setting

$$w(\pm z) = \phi_{(3z-1)s}(w(\pm 1/3)), \qquad \forall z \in [1/3, 2/3].$$

Notice that $w(\pm z) \in U_\pm$ for all $z \in (0, 2/3]$, and $E_k(w(\pm 2/3)) < \delta^2$. We set

$$\boldsymbol{y}^\pm = (y_0^\pm, ..., y_{k-1}^\pm) := w(\pm 2/3).$$

For each $r \in [0,1]$, we define $\boldsymbol{y}^\pm(r) = (y_0^\pm(r), ..., y_{k-1}^\pm(r))$ by

$$y_i^\pm(r) := \exp_{y_0^\pm}((1-r)\exp_{y_0^\pm}^{-1}(y_i^\pm)).$$

Notice that $\boldsymbol{y}^\pm(0) = \boldsymbol{y}^\pm$, $\boldsymbol{y}^\pm(r) \in U_\pm$, and

$$E_k(\boldsymbol{y}^\pm(r)) = k \sum_{i \in \mathbb{Z}_k} d(y_i^\pm(r), y_{i+1}^\pm(r))^2 < 4k^2\delta^2 \leq \mathrm{sys}(M,g)^2, \qquad \forall r \in [0,1],$$

$$E_k(\boldsymbol{y}^\pm(1)) = 0.$$

We extend w to a continuous map $w : [-1,1] \to \Lambda_k M$ by setting

$$w(\pm z) = \boldsymbol{y}^\pm(3z-2), \qquad \forall z \in [2/3, 1].$$

Finally, we define $u := \iota \circ w : [-1,1] \to \Lambda M$. Notice that the associated continuous map $\tilde u : S^2 \to M$ has degree 1; indeed, the preimage $u^{-1}(\gamma(t))$ is a singleton for every $t \in S^1$, and the restriction of u to a neighborhood of $u^{-1}(\gamma)$ is a homeomorphism onto its image. Therefore $u \in \mathcal{U}$, and

$$\mathrm{bir}(M,g) \leq \max_{z \in [-1,1]} E(u(z))^{1/2} = E(u(0))^{1/2} = \mathrm{sys}(M,g).$$

On the other hand, $\mathrm{bir}(M,g)^2$ is a positive critical value of E, and therefore

$$\mathrm{bir}(M,g) \geq \mathrm{sys}(M,g).$$

□

Proof of Theorem 0.1. We set $M_i := \partial B_i$, $i = 1, 2$. Since the regions $B_1 \subset B_2$ are strictly convex, for each $x \in M_2$ there exists a unique $\pi(x) \in M_1$ such that

$$\|x - \pi(x)\| = \min_{y \in M_1} \|x - y\|.$$

The map $\pi : M_2 \to M_1$ is a 1-Lipschitz homeomorphism with respect to the Riemannian metrics g_i on M_i that are restriction of the ambient Euclidean metric. In particular, for every $W^{1,2}$ curve $\gamma_2 : S^1 \to M_2$, if we denote by $\gamma_1 := \pi \circ \gamma_2$ its image in M_1, we have

$$\int_{S_1} \|\dot\gamma_2(t)\|^2 \mathrm{d}t \geq \int_{S_1} \|\dot\gamma_1(t)\|^2 \mathrm{d}t$$

We denote by $\mathcal{U}_1$ and $\mathcal{U}_2$ the family of maps involved in the definition of the Birkhoff min-max values of M_1 and M_2 respectively. Notice that $\pi \circ u \in \mathcal{U}_1$ for all $u \in \mathcal{U}_2$. Therefore, if we denote the energy of $W^{1,2}$ loops $\gamma : S^1 \to \mathbb{R}^3$ by

$$E(\gamma) = \int_{S^1} \|\dot\gamma(t)\|^2 \, \mathrm{d}t,$$

we have

$$\mathrm{bir}(M_2) = \inf_{u \in \mathcal{U}_2} \max_{z \in [-1,1]} E(u(z))^{1/2} \geq \inf_{u \in \mathcal{U}_2} \max_{z \in [-1,1]} E(\pi \circ u(z))^{1/2} \geq \mathrm{bir}(M_1).$$

This, together with Lemma 0.2, implies that $\mathrm{sys}(M_2) \geq \mathrm{sys}(M_1)$. □

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