A FEW PROPERTIES OF BESSE CONTACT MFDS

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Joint work with

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- (Y^{2n-1}, λ) closed, connected contact mfd

- $\phi_\lambda^t \; Y \circlearrowleft$ Reeb flow

$\lambda \wedge d\lambda^{n-1}$ volume form

$$\frac{d}{dt}\phi_\lambda^t = R_\lambda \circ \phi_\lambda^t$$

$$\begin{cases} \lambda(R_\lambda) \equiv 1 \\ d\lambda(R_\lambda, \cdot) \equiv 0 \end{cases}$$

Def (Y, λ) is BESSE when all its Reeb orbits are closed

example Rational ellipsoids

$$E = E(a_1, .., a_n) = \left\{ z \in \mathbb{C}^n \,\middle|\, \sum_j \frac{|z_j|^2}{a_j} \leq \frac{1}{\pi} \right\}$$

$$\lambda = \frac{1}{4} \sum_j \left(z_j d\bar{z}_j - \bar{z}_j d z_j \right)$$ contact form on ∂E

$$\phi_\lambda^t (z_1, .., z_n) = \left(e^{i 2\pi t / a_1} z_1, \; . \; , \; e^{i 2\pi t / a_n} z_n \right)$$

$a_j / a_k \in \mathbb{Q} \quad \forall j, k$

Wadsley thm A Besse Reeb flow is periodic,

i.e. $\phi_\lambda^T = \mathrm{id}$ for some $T > 0$

Rmk Wadsley thm does NOT hold for general (non Reeb) flows all of whose orbits are closed (Sullivan)

Def (Y, λ) is ZOLL when every Reeb orbit has minimal period $T > 0$

example
- Round sphere $S^{2n-1} \subset \mathbb{C}^n$
- Unit cotangent bundle of round n-spheres

REMARKABLE PROPERTIES OF BESSE & ZOLL CONTACT FORMS

Thm (Cristofaro Gardiner, Mazzucchelli)

- $\underbrace{(Y^3, \lambda)}_{\text{closed, connected}}$ is Besse iff $\underbrace{\sigma(Y, \lambda)}_{\text{action spectrum}} \subset a\mathbb{Z}$ for some $a > 0$
- Two Besse contact forms λ_1, λ_2 on Y^3 are strictly contactomorphic iff $\underbrace{\sigma_p(Y, \lambda_1)}_{\text{prime action spectrum}} = \sigma_p(Y, \lambda_2)$

Combining with the classification of Seifert fibration of Geiges-Lange, we obtain

Cor Any Besse (S^3, λ) is strictly contactomorphic to a rational ellipsoid

Q Does this hold in higher dimension?

This is an open question even for Zoll spheres.
A related open question is the uniqueness of symplectic forms on $\mathbb{C}P^n$

STRUCTURE OF A BESSE CONTACT MFD (Y, λ)

- $\forall\; a \in \sigma(Y,\lambda)$

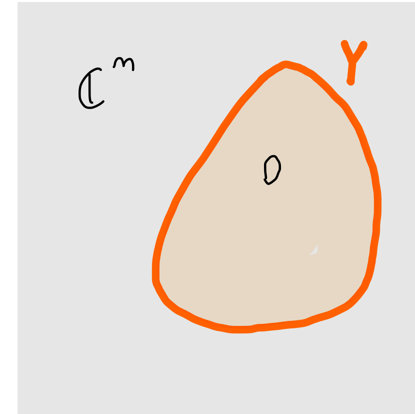
  $$Y_a = \mathrm{fix}\left(\phi_\lambda^a\right)$$ (subspace of a-periodic Reeb orbits)

  Every connected component of Y_a is a contact submanifold of (Y,λ)

- Morse-Bott property

  $$\dim(Y_a) = \dim \mathrm{Ker}\left(d\phi_\lambda^a(z) - I\right) \quad \forall z \in Y_a$$

- **Def** (Y,λ) is a convex contact sphere when



$$\begin{cases} Y \subset \mathbb{C}^n \text{ hypersurface } \cong S^{2n-1} \\ \qquad \text{with positive curvature} \\ 0 \text{ enclosed by } Y \\ \lambda = \frac{i}{4} \sum_j \left(z_j \, d\bar{z}_j - \bar{z}_j \, dz_j \right) \end{cases}$$

We do not know if Besse convex contact spheres are necessarily rational ellipsoids, but at least they resemble rational ellipsoids

Thm (Mazzucchelli-Radeschi)

Let (Y,λ) be a Besse convex contact sphere

Then every non-empty subspace $Y_a = \mathrm{fix}(\phi_\lambda^a)$

is an integral homology sphere

Proof (inspired by Radeschi-Wilking's recent proof of Berger conjecture for Riemannian S^n)

- Clarke action functional $\Psi \quad \underset{\substack{\circlearrowleft \\ S^1}}{\Lambda} \to (0,\infty)$

$$\mathrm{Crit}(\Psi) \cap \Psi^{-1}(a) \cong Y_a$$

- Morse indices of Ψ = Maslov indices

  Convexity of Y make indices grow under iteration of closed Reeb orbits

- $\Lambda \cong \underbrace{S^\infty}_{\text{sphere of separable Hilbert space}}$ S^1-equivariant homotopy equivalence

- Negative eigenspaces of critical mfds of Ψ are orientable

- Ψ is perfect for the S^1-equivariant Morse theory $\square$

$B \subset \mathbb{C}^n$ compact

$k = 1, 2, 3, \ldots$

$c_k(B)$ = k-th Ekeland-Hofer capacity

Properties

- c_k are symplectic invariants, monotone under inclusion, 1-homogeneous
- c_1 is an ordinary symplectic capacity

$$c_1(B^{2n}) = c_1(B^2 \times \mathbb{R}^{2n-2}) = \pi$$

- If $B \subset \mathbb{C}^n$ is a compact domain with smooth restricted contact type boundary $(\partial B, \lambda)$, then

$$\boxed{c_k(B) = c_k(\partial B) \in \sigma(\partial B, \lambda)}$$

If B is convex, then

$$\boxed{c_1(B) = c_1(\partial B) = \underbrace{\min(\sigma(\partial B, \lambda))}_{\text{systole}}}$$

- $0 < a_1 \leq a_2 \leq \ldots \leq a_m$

$\sigma(\partial E(a_1, \ldots, a_m))$ = multiples of the a_j's

We enumerate $\sigma(\partial E(a_1, \ldots, a_m))$ in increasing order as $\sigma_1 < \sigma_2 < \sigma_3 < \ldots$

$c_k(E(a_1, \ldots, a_m))$ = k-th element of the sequence

$$\underbrace{\sigma_1, \ldots, \sigma_1}_{\times d_1}, \underbrace{\sigma_2, \ldots, \sigma_2}_{\times d_2}, \underbrace{\sigma_3, \ldots, \sigma_3}_{\times d_3}, \ldots$$

where $d_j = \# \{ i \mid a_i \text{ divides } \sigma_j \}$

Thm (Mazzucchelli–Radeschi)

Let (Y,λ) be a Besse convex contact sphere, $\sigma_1 < \sigma_2 < \sigma_3 < \dots$ the elements of $\sigma(Y,\lambda)$

Then $c_K(Y)$ is the K-th element of the sequence

$$\underbrace{\sigma_1, \dots, \sigma_1}_{\times d_1}, \underbrace{\sigma_2, \dots, \sigma_2}_{\times d_2}, \underbrace{\sigma_3, \dots, \sigma_3}_{\times d_3}, \dots$$

where $\dim \operatorname{fix}(\phi_\lambda^{\sigma_J}) = 2d_J - 1$

Q Do the two previous thms hold for non-convex Besse contact spheres ?

A GENERALIZATION OF VITERBO CONJECTURE

Viterbo conjecture (2000)

Every convex body $C \subset \mathbb{C}^n$ satisfies

$$\frac{c_1(C)}{\mathrm{vol}(C)^{1/n}} \leq 1$$

with equality iff $C \cong B^{2n}$

symplectomorphic

k-th capacity ratio

$$\mathcal{S}_k(C) = \frac{c_k(C)}{\mathrm{vol}(C)^{1/m}}, \qquad C \subset \mathbb{R}^{2m} \text{ convex}$$

Rmk (Abbondandolo-Lange-Mazzucchelli)

Consider the function $(a_1, \ \ , a_m) \mapsto \mathcal{S}_k(E(a_1, \dots, a_m))$

i) Its local maximizers are those vectors s.t.

$$r_1 a_1 = r_2 a_2 = . \ \ = r_m a_m$$

for some positive integers s.t.

$$r_1 + \ \ + r_m = k + m - 1$$

Rmk (Abbondandolo-Lange-Mazzucchelli)

ii) Consider the function $(a_1, \ , a_m) \mapsto S_K(E(a_1, .. , a_m))$

$K = qm + r$, where $q = \lceil \frac{K}{m} \rceil - 1$

Its global maximum is

$$(q+1)^{\frac{m-r+1}{m}} (q+2)^{\frac{r-1}{m}}$$

and it is achieved precisely at those vectors of the form

$$a_m = a_{m-1} = .. = a_r$$

$$a_{r-1} = a_{r-2} = \ . \ = a_1 = \frac{q+1}{q+2} a_m$$

Question

Let $K = qn + r$, where $q = \lceil \frac{K}{n} \rceil - 1$

If $C \subset \mathbb{R}^{2n}$ convex, is it true that

$$S_K(C) \leq (q+1)^{\frac{n-r+1}{n}} (q+2)^{\frac{r-1}{n}}$$

with $=$ iff C is symplectomorphic to the rational ellipsoid of remark ,, ?

(For $K=1$, this is Viterbo conjecture)

Thm (Abbondandolo-Bramham-Hrynewicz-Salomao)

$n=2, \ k=1$

Local maximizers* of $\underbrace{C}_{\substack{\text{convex}\\ \text{in } \mathbb{R}^4}} \mapsto S_1(C)$ are those C symplectomorphic to a round ball $B^4 \subset \mathbb{R}^4$

In higher dimension

Thm (Abbondandolo-Benedetti)

The round balls $B^{2n} \subset \mathbb{R}^{2n}$ are local maximizers* of $C \mapsto S_1(C)$

* with the C^3-topology on the space of convex bodies

Both thms follow from contact systolic statements

(Y^{2n-1}, λ) closed, connected, contact mfd

$T_1(\lambda)$ = minimal Reeb period = $\min \sigma(Y, \lambda)$

$$S_1(\lambda) = \frac{T_1(\lambda)}{\mathrm{vol}(Y,\lambda)^{1/n}}$$ systolic ratio

Thm (Abbondandolo-Benedetti)

The C^3-local maxima of S_1 are the Zoll contact forms

Goal Characterize Besse contact forms as local maxima of suitable higher "systolic" ratio

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(Y^3, λ) closed, connected, contact 3-manifold

$k = 1, 2, 3, \ldots$

$$\tau_k(\lambda) = \min \left\{ \tau > 0 \;\middle|\; \sum_{0 < t \leq \tau} \# \{t\text{-periodic Reeb orbits}\} \geq k \right\}$$

$$S_k(\lambda) = \frac{\tau_k(\lambda)}{\mathrm{vol}(Y, \lambda)^{1/2}}$$

Let (Y^3, λ) be Besse

$\bar{K}(\lambda)$ = minimal positive integer K s.t.
$T_K(\lambda)$ is the minimal common period of the closed Reeb orbits

$$0 < T_1(\lambda) < \ldots < T_{\bar{K}(\lambda)}(\lambda) = T_{\bar{K}(\lambda)+1}(\lambda) = T_{\bar{K}(\lambda)+2}(\lambda) = \ldots$$

Rmk Every C^∞-local maximizer of $\lambda \mapsto S_K(\lambda)$ is a Besse contact form λ s.t. $\bar{K}(\lambda) \leq K$

(if $Y = S^3$ then $\bar{K}(\lambda) = K$)

Thm (Abbondandolo-Lange-Mazzucchelli)

Let (Y^3, λ_0) be a Besse contact mfd, $K = \bar{K}(\lambda_0)$

Then $\exists$ C^3-open nbhd U of λ_0 s.t.

$$S_K(\lambda) \leq S_K(\lambda_0) \quad \forall \lambda \in U$$

$=$ iff $\psi^* \lambda = c\lambda_0$ for some diffeo $\psi : Y \circlearrowleft$ and constant $c > 0$

Proof (builds on and extend the $K=1$ case (Zoll), which was proven by Abbo.-Bramham-Hryniewicz-Salomao for S^3, and Benedetti-Kang in general)

- (Y^3, λ_0) Besse, $K = \bar{K}(\lambda_0)$, wlog $\tau_K(\lambda_0) = 1$
- λ C^3-close to λ_0, wlog $\tau_K(\lambda) = 1$

Goal $\mathrm{vol}(Y, \lambda) \geq \mathrm{vol}(Y, \lambda_0)$

- wlog $\exists\, \gamma$ 1-periodic Reeb orbit for both λ and λ_0
- γ has minimal period $\frac{1}{m}$, for some integer $m \geq 1$

- $\exists$ "global surface of section" for λ_0 with boundary on γ

$$\underset{\text{compact surface}}{\Sigma} \xrightarrow{f} Y \qquad \text{s.t.}$$

1) $\operatorname{int}(\Sigma) \xhookrightarrow{f} Y \setminus \gamma$ embedding $\pitchfork$ Reeb R_{λ_0}, every Reeb orbit with minimal period 1 intersects the image of f in q points

2) $\partial\Sigma \xrightarrow[p \,:\, 1]{f} \gamma$ covering map

3) $\mathrm{vol}(Y, \lambda_0) = -$ Euler number of $(Y, \lambda_0) = \dfrac{p}{m \quad q}$

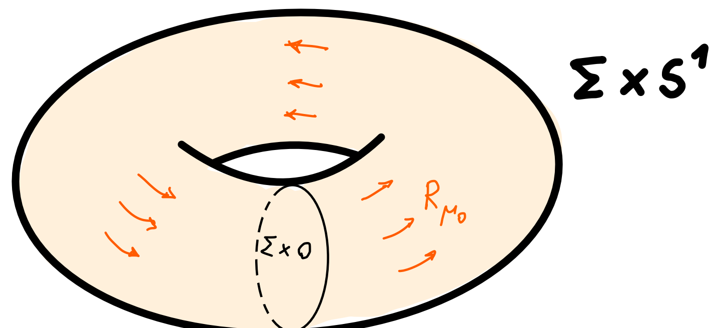
- "open book decomposition" of Y

$$\begin{aligned} \Sigma \times S^1 &\xrightarrow{F} Y \\ (z,t) &\longmapsto \phi^t_{\lambda_0}(f(z)) \end{aligned} \qquad (S^1 = \mathbb{R}/\mathbb{Z})$$

$$\operatorname{int}(\Sigma) \times S^1 \xrightarrow[p \;\; 1]{F} Y \setminus \gamma \qquad \text{covering map}$$

$$\mu_0 = F^* \lambda_0$$

$$R_{\mu_0} = \partial_t$$



- (remember that $R_\lambda|_\gamma = R_{\lambda_0}|_\gamma$, and $\lambda - \lambda_0$ is C^3 small)

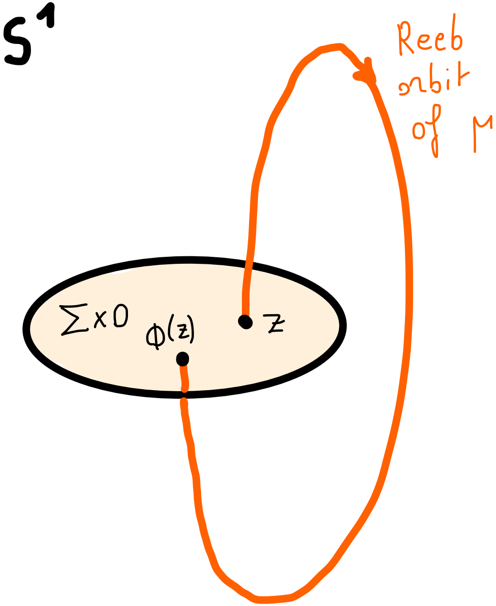
$\mu := F^*\lambda$ contact form on $\mathrm{int}(\Sigma) \times S^1$

R_μ is C^1-close to $R_{\mu_0} = \partial_t$,
extends smoothly to $\Sigma \times S^1$

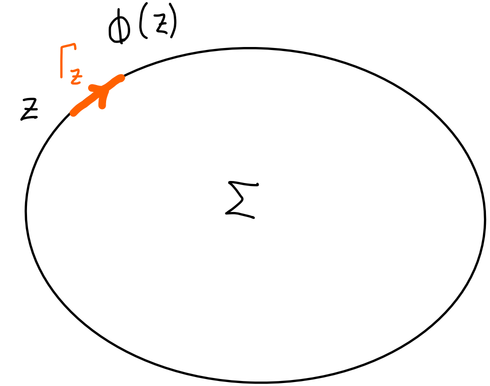
$\tau : \Sigma \to (0,\infty)$ 1-st return time to $\Sigma \times 0$

$\phi : \Sigma \circlearrowleft \cong$ 1-st return map

$(\phi(z), 0) = \phi_\mu^{\tau(z)}(z, 0)$



$\sigma = \tau - 1$ is C^1-small



$$\sigma(z) = \int_{\Gamma_z} \mu \qquad \forall z \in \partial\Sigma$$

ϕ is C^1-close to id

- $\nu := \mu|_{\Sigma \times 0} \quad \Rightarrow \quad d\nu$ symplectic on $\mathrm{int}(\Sigma)$

$$\phi^*\nu - \nu = d\tau = d\sigma$$

$$\int_\Sigma \sigma \, d\nu = \int_\Sigma \tau \, d\nu + \int_\Sigma d\nu = q \, \mathrm{vol}(Y, \lambda) + \frac{p}{m}$$

$$= q \left(\mathrm{vol}(Y, \lambda) - \mathrm{vol}(Y, \lambda_0) \right)$$

If $\mathrm{vol}(Y,\lambda) < \mathrm{vol}(Y,\lambda_0)$

then $\displaystyle\int_\Sigma \sigma \, dv < 0$, and a fixed point thm

implies

$$\exists\, z \in \mathrm{fix}(\phi) \text{ with } \sigma(z) < 0$$

$$\tau(z) < 1 = \tau_k(\lambda)$$

But $\tau \sim 1$, therefore $\tau(z) > \tau_{k-1}(\lambda)$

$\tau(z) \in \left(\tau_{k-1}(\lambda), \tau_k(\lambda)\right)$ $\lightning$ $\square$