C^2 structurally stable Riemannian geodesic flows of closed surfaces are Anosov

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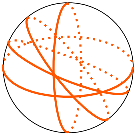
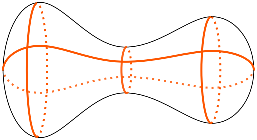
Joint work with Gonzalo Contreras

Closed geodesics of Riemannian manifolds





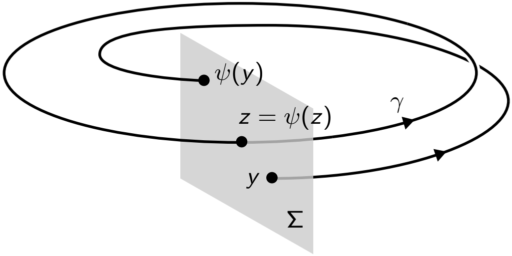
The closed geodesics of (M, g) are the periodic orbits of the geodesic flow

$\phi_t : SM \to SM, \quad \phi_t(\gamma(0)) = \gamma(t)$
where $\gamma = (x, \dot{x})$, and $x : \mathbb{R} \to M$ is a geodesic with $\|\dot{x}\|_g = 1$.

Closed geodesics of Riemannian manifolds

$\Sigma \subset SM$ cross section at a closed geodesic γ

$\psi : \Sigma \to \Sigma$, $\psi(y) = \phi_{\tau(y)}(y)$ first-return map

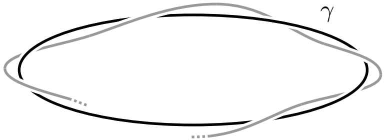


The Floquet multipliers of γ are the eigenvalues of $d\psi(z)$.

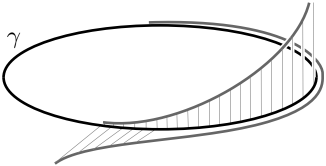
Closed geodesics of Riemannian manifolds

A closed geodesic γ is

- elliptic when its Floquet multipliers are in $S^1 \subset \mathbb{C}$



- hyperbolic when its Floquet multipliers are in $\mathbb{R} \setminus \{1, -1\}$



- non-degenerate when its Floquet multipliers are in $\mathbb{C} \setminus \{1\}$.

Some history

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- (Contreras-Oliveira, 2004) A C^2 generic Riemannian metric on S^2 has an elliptic closed geodesic.

Hyperbolicity

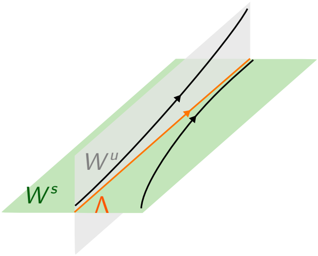
$\phi_t : N \to N$ flow of a vector field X

A compact invariant subset $\Lambda \subseteq N$ is hyperbolic when there exists a ϕ_t-invariant splitting

$$TN|_\Lambda = E^s \oplus E^u \oplus \mathrm{span}\{X\}$$

such that, for some $b, c > 0$,

- $\|d\phi_t \cdot v\| \leq b\, e^{-ct}\|v\|$ for all $v \in E^s$, $t \geq 0$,
- $\|d\phi_{-t} \cdot v\| \leq b\, e^{-ct}\|v\|$ for all $v \in E^u$, $t \geq 0$.



$\phi_t : N \to N$ is Anosov when the whole N is hyperbolic.

Main result

Theorem (Contreras-Mazzucchelli) *On any closed surface, there exists a C^2-open and dense subset $\mathcal{U}$ of smooth Riemannian metrics such that, for each $g \in \mathcal{U}$, the associated geodesic flow is Anosov or has an elliptic closed orbit.*

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Remark. Surfaces of genus ≤ 1 do not admit Anosov geodesic flows (Margulis). Therefore, for these surfaces, each $g \in \mathcal{U}$ has an elliptic closed orbit.

Consequences

A geodesic flow $\phi_t^{g_0}$ is C^2-structurally stable when g_0 has a C^2-open neighborhood $\mathcal{V}$ and, for each $g_1 \in \mathcal{V}$, there is a homeohorphism

$$\kappa : S^{g_0}M \to S^{g_1}M$$

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Remark. Anosov himself showed that Anosov geodesic flows are C^2-structurally stable.

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- Assume that $\phi_t^{g_0}$ is structurally stable within the neighborhood $\mathcal{V}$ of g_0.
- A geodesic flow with an elliptic periodic orbit is not C^2-structurally stable. Therefore no $g \in \mathcal{V}$ has elliptic closed geodesics.

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- By the structural stability, $\overline{\mathrm{Per}(\phi_t^{g_0})} \cong \overline{\mathrm{Per}(\phi_t^g)} = S^g M$. Therefore $\phi_t^{g_0}$ is Anosov. $\square$

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A geodesic flow $\phi_t^{g_0}$ is C^2-stably ergodic when g_0 has a C^2-open neighborhood $\mathcal{W}$ and, for each $g_1 \in \mathcal{W}$, the geodesic flow $\phi_t^{g_1}$ is ergodic: its invariant subsets have either full measure or zero measure.

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Theorem (Knieper, Schulz). *A C^2-stably ergodic geodesic flow of a closed surface must be Anosov.*

Reeb flows

Closed contact manifold:

N closed manifold of odd dimension $2n+1$
λ 1-form on N such that $\lambda \wedge (d\lambda)^n$ is nowhere vanishing

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The Reeb vector field X of (N, λ) is defined by

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Geodesic flows of Riemannian manifolds (M, g) are examples of Reeb flows:

$N = SM = \{(x, v) \in TM \mid \|v\|_g = 1\}$
$\lambda_{(x,v)} = g(v, d\pi(x, v) \cdot)$, where $\pi : SM \to M$, $\pi(x, v) = x$.

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Surfaces of section

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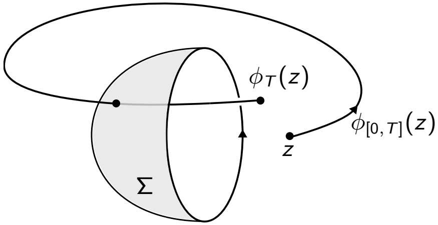
i.e. an immersed compact surface $\Sigma \subset N$ such that

- $\mathrm{int}(\Sigma)$ is injectively immersed and transverse to the Reeb vector field X
- $\partial\Sigma$ is a union of closed Reeb orbits
- There exists $T > 0$ such that every segment of Reeb orbit $\phi_{[0,T]}(z)$ intersects Σ.



Broken book decompositions

Long-standing open question. *Does any closed contact 3-manifold (N, λ) admit a global surface of section for its Reeb flow ϕ_t?*

Known answers:

- (Fried 1981) Yes if ϕ_t is Anosov.
- (Hofer-Wysocki-Zehnder 1998) Yes if (N, λ) is a convex 3-sphere.

Broken book decompositions (Colin-Dehornoy-Rechtman)

A broken book decomposition of (N^3, λ) is given by:

- A family of pages $\mathcal{F}$. Each page $\Sigma \subset \mathcal{F}$ is a (not necessarily global) surface of section for the Reeb flow.
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- There exists finitely many pages $\Sigma_1, ..., \Sigma_n$ such that:
  - Every Reeb orbit $t \mapsto \phi_t(z)$ intersects $\Sigma_1 \cup ... \cup \Sigma_n$.
  - If $\phi_{[0,\infty)}(z) \notin \Sigma_1 \cup ... \cup \Sigma_n$, then $z \in W^s(\gamma)$ for some $\gamma \subset K$.
  - If $\phi_{(-\infty,0]}(z) \notin \Sigma_1 \cup ... \cup \Sigma_n$, then $z \in W^u(\gamma)$ for some $\gamma \subset K$.

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The proof requires Hutchings' embedded contact homology, which provides surfaces of section through any given point of the contact manifold N as projections of suitable holomorphic curves in the symplectization $\mathbb{R} \times N$.

A characterization of Anosov Reeb flows

Theorem (Contreras-Mazzucchelli). *Let (N, λ) be a closed contact 3-manifold such that:*

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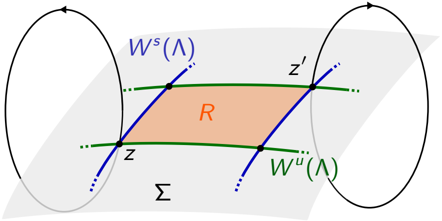
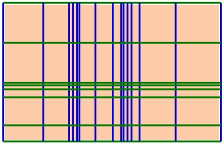
- We fix a small heteroclinic rectangle $R \subset \mathrm{int}(\Sigma)$:



$z, z' \in \Lambda$

A characterization of Anosov Reeb flows

- $R \cap \left(W^s(\Lambda) \cup W^u(\Lambda)\right)$ is compact and connected



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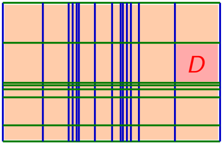

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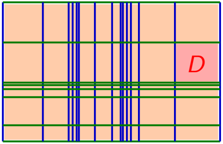

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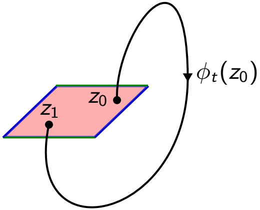
- (Poincaré recurrence) $\exists z_0 \in D,\ t_0 > 0$ such that $z_1 := \phi_{t_0}(z_0) \in D$.



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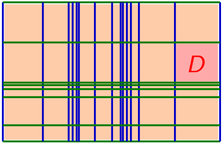

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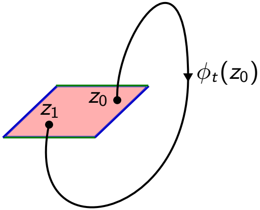

- We extend the map $z_0 \mapsto z_1$ to a smooth return map $\psi : U \to \Sigma$ on a maximal open subset $U \subseteq D$.

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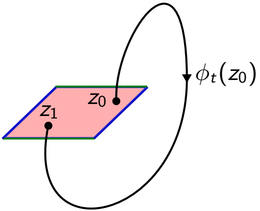

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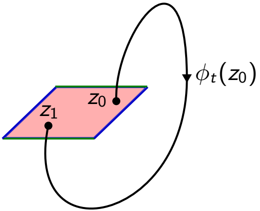

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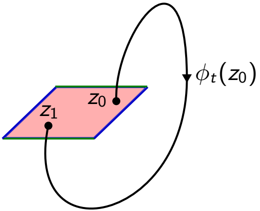

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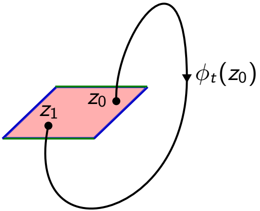

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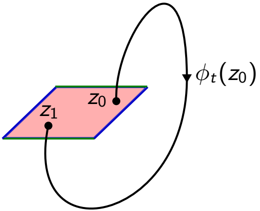

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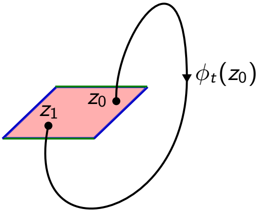

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- Thus $z \in D \cap \mathrm{Per}(X)$. But $D \cap \mathrm{Per}(X) \subset D \cap \Lambda = \varnothing$. $\square$

Thank you for your attention!