Existence of global surfaces of section for Kupka-Smale Reeb vector fields of 3-dimensional closed contact manifolds

Marco Mazzucchelli
(CNRS and École normale supérieure de Lyon)

Joint work with Gonzalo Contreras

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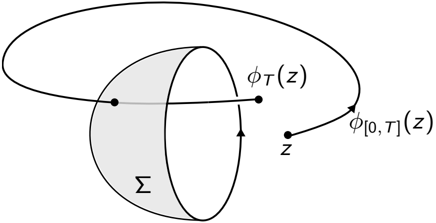
A global surface of section is a compact immersed surface $\Sigma \looparrowright N$ such that:

- $\mathrm{int}(\Sigma)$ is embedded and transverse to X,
- $\partial\Sigma$ is tangent to X,
- for some $T > 0$, any orbit segment $\phi_{[0,T]}(z)$ intersects Σ.



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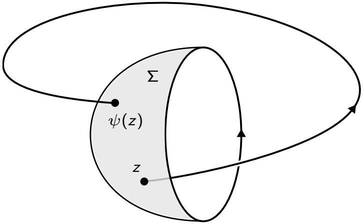

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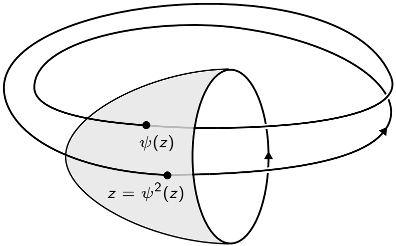
$$\psi : \mathrm{int}(\Sigma) \to \mathrm{int}(\Sigma), \quad \psi(z) = \phi_{\tau(z)}(z)$$

Remark. $\mathrm{Per}(\psi) \xleftrightarrow{1:1} \mathrm{Per}(X) \cap \mathrm{int}(\Sigma)$



Global surfaces of section of Reebs flows

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$\lambda \wedge d\lambda$ volume form
$\lambda(X) \equiv 1, \;\; d\lambda(X, \cdot) \equiv 0$

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$N = SM$,
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Remarks.

- $d\lambda|_{\Sigma}$ is an area form, and therefore $\partial\Sigma \neq \varnothing$
- The first return map $\psi : \mathrm{int}(\Sigma) \to \mathrm{int}(\Sigma)$ preserves $d\lambda$, and indeed $\psi^*\lambda = \lambda + d\tau$

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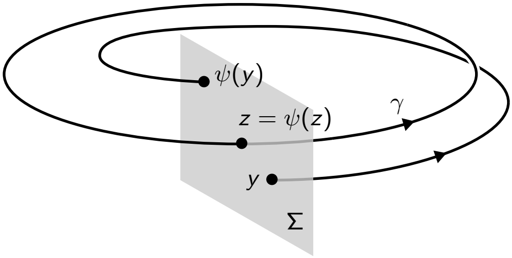
  An application: any contact convex 3-spheres has either exactly two or infinitely many closed Reeb orbits

Closed Reeb orbits

$\Sigma \subset N$ cross section at a closed Reeb orbit γ

$\psi : \Sigma \to \Sigma$, $\psi(y) = \phi_{\tau(y)}(y)$ first-return map



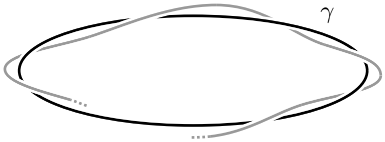
The Floquet multipliers of γ are the eigenvalues of $d\psi(z)$:

$$\sigma(d\psi(z)) = \{\lambda, \lambda^{-1}\} \subset S^1 \cup \mathbb{R}.$$

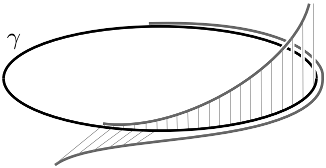
Closed Reeb orbits

The closed Reeb orbit γ is

- elliptic when its Floquet multipliers are in $S^1 \subset \mathbb{C}$



- hyperbolic when its Floquet multipliers are in $\mathbb{R} \setminus \{1, -1\}$



- non-degenerate when its Floquet multipliers are not complex roots of 1.

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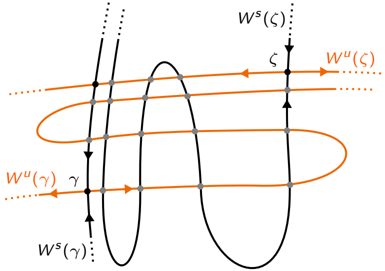

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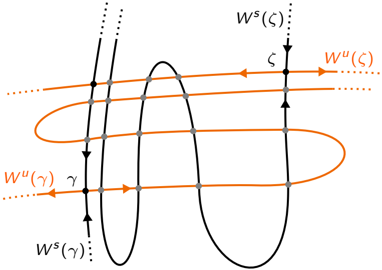

Remarks. Kupka-Smale holds for:

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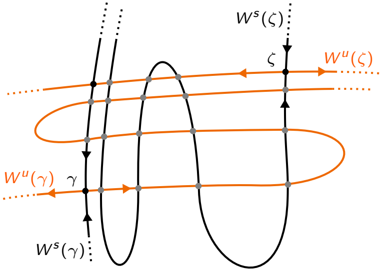

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- (Robinson) the Reeb vector field of a C^∞ generic contact form on a closed 3-manifold;
- (Contreras-Paternain) the geodesic vector field of a C^∞ generic Riemannian metric on a closed surface.

Main theorem

Theorem (Contreras-Mazzucchelli). *Any Kupka-Smale Reeb vector field on a closed 3-manifold admits a global surface of section.*

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Remarks.

- Independently, Colin-Dehornoy-Hryniewicz-Rechtman proved the existence of a global surface of section for the Reeb vector field of a closed 3-manifold, provided there exists a suitable cohomology class integrating positively on a suitable collection of periodic orbits.
  Together with Irie's equidistribution theorem, this gives an alternative argument for the above Corollary (i).

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Remarks.

- The existence of a global surface of section with non-degenerate boundary is a C^1-open condition in the vector field (thus a C^2-open condition in the contact form)
- The Kupka-Smale condition in the above theorem is only required on a suitable finite collection of hyperbolic periodic orbits.

Broken book decompositions (Colin-Dehornoy-Rechtman)

A broken book decomposition of (N^3, λ) is given by:

- A family of pages $\mathcal{F}$. Each page $\Sigma \subset \mathcal{F}$ is a (not necessarily global) surface of section for the Reeb flow.
- The binding $K = K_{\text{rad}} \cup K_{\text{br}} = \bigcup_{\Sigma \in \mathcal{F}} \partial\Sigma$, which is a finite link.

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or broken:

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- There exists finitely many pages $\Sigma_1, ..., \Sigma_n$ such that:
  - Every Reeb orbit $t \mapsto \phi_t(z)$ intersects $\Sigma_1 \cup ... \cup \Sigma_n$.
  - If $\phi_{[0,\infty)}(z) \notin \Sigma_1 \cup ... \cup \Sigma_n$, then $z \in W^s(K_{\mathrm{br}})$.
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Remark. Broken book decompositions are a generalization of Hofer-Wysocky-Zehnder's finite energy foliations.

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The proof requires Hutchings' embedded contact homology, which provides surfaces of section through any given point of the contact manifold N as projections of suitable holomorphic curves in the symplectization $\mathbb{R} \times N$.

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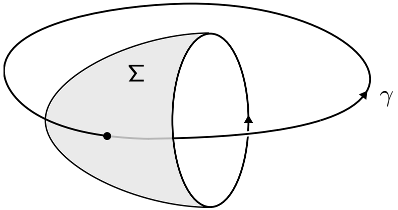

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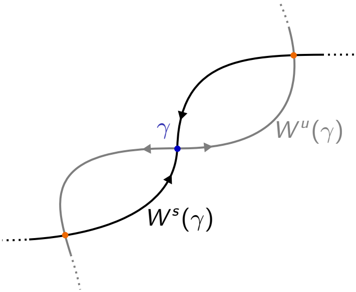

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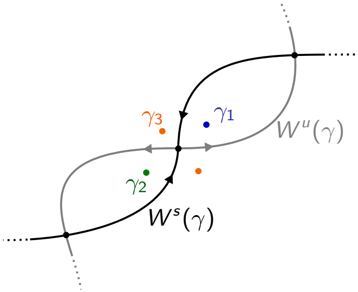

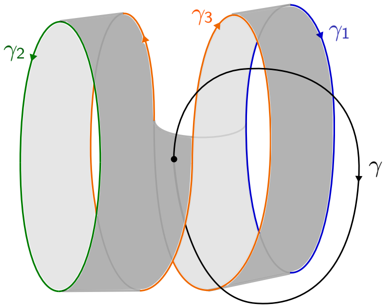

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In order to find a surface of section, we have to show that there is always some broken binding component $\gamma \subset K_{\mathrm{br}}$ with homoclinics in all separatrices.

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Theorem (Contreras-Mazzucchelli) *Let* (N, λ) *be a Kupka-Smale closed contact 3-manifold, with a broken book decomposition. Any broken binding component has homoclinics in all separatrices.*

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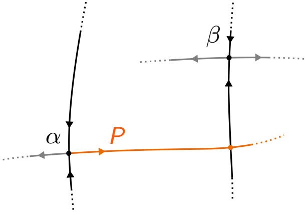

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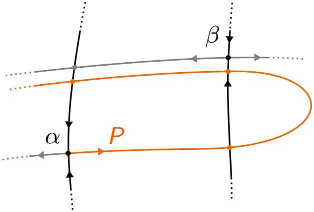

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- (Hofer-Wysocki-Zehnder) For every $\alpha \subset K_{\mathrm{br}}$, every connected component $P \subset W^u(\alpha) \setminus \alpha$ satisfies $P \cap W^s(K_{\mathrm{br}}) \neq \varnothing$.

From broken books to rational open books

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Proof

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$$\gamma_{-1} \rightsquigarrow \gamma \rightsquigarrow \gamma_1$$

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  i.e. γ has homoclinics

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  $\alpha \rightsquigarrow \ldots \rightsquigarrow \alpha \rightsquigarrow \gamma_{-2} \rightsquigarrow \gamma_{-1} \rightsquigarrow \gamma \rightsquigarrow \gamma_1 \rightsquigarrow \gamma_2 \rightsquigarrow \ldots \rightsquigarrow \beta \rightsquigarrow \ldots \rightsquigarrow \beta$

  Therefore $\beta \rightsquigarrow \alpha$ and $\gamma \rightsquigarrow \beta \rightsquigarrow \alpha \rightsquigarrow \gamma$

  i.e. γ has homoclinics in every separatrix. □

An application: Anosov Reeb flows

- Using HWZ's finite energy foliations, Contreras-Oliveira established the following outstanding result, generalizing a (not entirely correct) claim by Poincaré:

  Theorem (Contreras-Oliveira, 2004) *A C^2-generic Riemannian metric on S^2 has an elliptic closed geodesic.*

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- Using broken book decompositions:

  Theorem (Contreras-Mazzucchelli). *Let (N, λ) be a closed contact 3-manifold with Reeb vector field X, such that:*
  - $\overline{\mathrm{Per}(X)}$ is hyperbolic,
  - $W^u(\alpha) \pitchfork W^s(\beta)$ for all $\alpha, \beta \subset \mathrm{Per}(X)$.

  Then X is Anosov.

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- Applying this theorem to geodesic flows:

  Theorem (Contreras-Mazzucchelli) *On any closed surface, there exists an C^2-open dense subset $\mathcal{U}$ of the space of Riemannian metrics such that any $g \in \mathcal{U}$ is Anosov or has an elliptic closed geodesic.*

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- This theorem provides a confirmation of the C^2-stability conjecture for Riemannian geodesic flows:

  Theorem (Contreras-Mazzucchelli) *The geodesic flow of a closed Riemannian surface is C^2-structurally stable if and only if it is Anosov.*

Thank you for your attention!