Section 2.8

Outline:

1. **Subspace**

   A subspace is NOT the same as "subset'.
   It has RULES (zero, addition, scalar multiplication).

2. **Column Space**

   Think about the column vectors of A.

3. **Null Space**

   Think about the solutions, $\vec{x}$, of A$\vec{x} = \vec{0}$.

4. **Basis**

   Think about sets of vectors and their linear combinations.

Subspace

Let H be a subset of $\Re^n$. It is a SUBSPACE of $\Re^n$ if it has the following properties:

(a) $\vec{0} \in H$.

(b) For EACH $\vec{u} \in H$ **and** $\vec{v} \in H$, **the sum** $\vec{u} + \vec{v} \in H$.

(c) For EACH $\vec{u} \in H$ **and scalar** c **the scalar product** $c\vec{u} \in H$.

CONSEQUENTLY:

For EACH $\vec{u} \in H$, $\vec{v} \in H$ **and scalars** c **and** k, **the linear combination,** $c\vec{u} + k\vec{v} \in H$.

Example 1

Are the following sets subspaces of $\Re^3$?

(a) The set of all vectors $\vec{b} \in \Re^2$**?**

(b) The set of all vectors $\vec{b} \in \Re^3 : b_1 = 2b_2$ **?**

(c) The set of all vectors $\vec{b} \in \Re^3 : b_1 = 3$ **?**

(d) The set of all vectors $\vec{b} \in \Re^3 : b_1 = 0$ **?**

Example

Let $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$**,** $\vec{v}_2 = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}$**, and** $\vec{w} = \begin{pmatrix} 11 \\ -2 \\ 1 \end{pmatrix}$**.**

Determine if $\vec{w}$ **is in the subspace of** $\Re^3$ **generated by** $\vec{v}_1$ **and** $\vec{v}_2$**.**

Equivalent ways of asking:

(a) **Determine if** $\vec{w}$ **is in the subspace of** $\Re^3$ **generated by** $\vec{v}_1$ **and** $\vec{v}_2$**.**

(b) **Is** $\vec{w}$ **in** $Span\{\vec{v}_1, \vec{v}_2\}$**?**

(c) **Can we write** $\vec{w}$ **as a linear combination of the vectors,** $\vec{v}_1$ **and** $\vec{v}_2$**?**

(d) **Are there scalars,** x_1 **and** x_2**, such that** $\vec{w} = x_1\vec{v}_1 + x_2\vec{v}_2$**?**

(e) **Is the system** $\mathbf{A}\vec{x} = \vec{w}$ **consistent, where** $\mathbf{A} = [\vec{v}_1 \;\; \vec{v}_2]$**?**

Column Space

Let $\mathbf{A} = [\vec{a}_1 \quad \vec{a}_2 \quad \cdots \quad \vec{a}_n]$. Then the column space of A, denoted by $ColA$, is the set of all linear combinations of the columns of A.

So, other ways of saying the same thing:

(a) $ColA$ **is the set of all vectors generated by the column vectors of A.**

(b) $ColA = Span\{\vec{a}_1, \vec{a}_2, ..., \vec{a}_n\}$**.**

(c) $ColA$ **is the set of all linear combination of the vectors,** $\vec{a}_1$**,** $\vec{a}_2$**, ... ,** $\vec{a}_n$**.**

(d) $ColA$ **is the set of all vectors** $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$**.**

(e) $ColA$ **is the set of all vectors** $\mathbf{A}\vec{x}$**.**

Example 1

Consider a matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \\ 1 & -5 & 2 \end{pmatrix} \text{ which is row equivalent to } \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{pmatrix}.$$

Which of the following set of vectors is in $ColA$?

(a) All vectors in $\Re^2$.
(b) All vectors in $\Re^3$.
(c) All vectors $\vec{b} \in \Re^3$ such that $b_1 + b_2 + 6b_3 = 0$.

Question 2

For all m x $n$matrices A, is $ColA$ necessarily a subspace of $\Re^m$?

Check the zero vector.
Check vectors with constraints.

Null Space

The null space of a matrix A, $NulA$, is the set of all solutions to $\mathbf{A}\vec{x} = \vec{0}$.

Example 1

Consider the matrix we saw before:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \\ 1 & -5 & 2 \end{pmatrix} \text{ row equivalent to } \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{pmatrix}.$$

Which of the following set of vectors is in $NulA$?

(a) All vectors in $\Re^2$.
(b) All vectors in $\Re^3$.
(c) All vectors $\vec{b} \in \Re^3$ such that $b_1 + b_2 + 6b_3 = 0$.
(d) The zero vector only.

Example 2

Consider the matrix before:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & 1 \\ 1 & -2 & -1 \end{pmatrix} \textbf{ row equivalent to } \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Which of the following set of vectors is in $NulA$?

(a) All vectors in $\Re^3$.

(b) $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

(c) All vectors $\vec{v} \in \Re^3$ such that $v_1 = v_2 = -v_3$.

(d) $\begin{pmatrix} 4 \\ 4 \\ -4 \end{pmatrix}$.

(e) $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

Question 2

For all m x n matrices A, is $NulA$ necessarily a subspace of $\Re^n$?

Basis

A basis for a subspace $H \in \Re^n$ is a linearly independent set in H that spans H.

Example 1

Consider this (standard) basis for $\Re^3$:

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Example 2

True or False:

If A is an nxn invertible matrix, then the column vectors of A form a basis for $\Re^n$.

Consider: Then A is row equivalent to the nxn identity matrix. And the columns of the identity matrix form the *standard* basis for $\Re^n$.

Example 3

Find a basis for the $ColA$ **if** $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$,

which is row equivalent to $\mathbf{R} = \begin{pmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Are the columns in the reduced echelon form (call them $\vec{r}_n$**,** $n = 1, 2, 3, 4, 5$**) linearly independent? Um, No. Do you agree that:**

$$\vec{r}_2 = 2\vec{r}_1; \qquad \vec{r}_5 = 3\vec{r}_1 - \vec{r}_3 + \vec{r}_4?$$

If yes, then any combination of $\vec{r}_1, ..., \vec{r}_5$ **is really a linear combination of** $\vec{r}_1, \vec{r}_3$**, and** $\vec{r}_4$**.**

So a basis for $ColA$ **is the set of** *pivot columns of* **A (and not the pivot columns of R):**

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$