Section 2.9: DIMENSION AND RANK

Outline:

1. **Coordinate Systems**

   You're all familiar with, say, the {x, y, z} coordinate system. We can use a *basis* to define other, more convenient coordinate systems.

2. **Dimension of a Subspace**

   Just the number of vectors you need in the basis.

3. **Rank**

   Dimension of the column space.

4. **Other Stuff**

   (a) **The Rank Theorem**
   (b) **The Basis Theorem**
   (c) **Even more of the Invertible Matrix Theorem.**

Coordinate Systems

Consider the $\{x, y, z\}$ coordinate system for $\Re^3$. It has basis vectors in those directions:

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Any vector $\vec{x}$ in $\Re^3$ can be represented as a linear sum of $\{\vec{i}, \vec{j}, \vec{k}\}$.

This notion generalizes: Consider a subspace H. And let $B = \{\vec{b}_1, \vec{b}_2, ..., \vec{b}_p\}$ be a basis for H. Then the coordinates of any vector $\vec{x}$ in H are the coefficients that describe $\vec{x}$ relative to this basis. So, the coordinates are the numbers $\{c_1, c_2, ..., c_p\}$, such that

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + ...c_p\vec{b}_p,$$

and we say that relative to the basis B,

$$[\vec{x}]_B = \begin{pmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_p \end{pmatrix}.$$

Note: the basis comprises linearly independent vectors. It has just the right amount of vectors you need - not too many (then they would be linearly dependent); not too few (then they would not allow for coordinates for all possible vectors.

Example 1 (#1)

Find the vector $\vec{x}$ determined by the given coordinate vector $[\vec{x}]_B$ and the given basis B.

$$B = \{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\}, \quad [\vec{x}]_B = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Example 2 (#6)

Find the vector $\vec{x}$ in a subspace H with a basis $B = \{\vec{b}_1, \vec{b}_2\}$. Find the B-coordinate vector of $\vec{x}$.

$$\vec{b}_1 = \begin{pmatrix} -3 \\ 1 \\ -4 \end{pmatrix}, \quad \vec{b}_2 = \begin{pmatrix} 7 \\ 5 \\ -6 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} 11 \\ 0 \\ 7 \end{pmatrix}.$$

Dimension of a Subspace

The **dimension** of a nonzero subspace H, dim$H$, is the number of vectors in any basis for H. The dimension of the zero subspace $\{\vec{0}\}$ is defined to be zero.

Example

Let H be the subspace of $\Re^3$ corresponding to Nul$A$ with

$$A = \begin{pmatrix} -3 & 7 & 11 \\ 1 & 5 & 0 \\ -4 & -6 & 7 \end{pmatrix} \text{ row equivalent to } \begin{pmatrix} 1 & 0 & -5/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}.$$

What is the dim$H$?

(Note - the dimension of a null space has to be the number of free variables in the null space.)

Rank

The **rank** of a matrix A, rank$A$, is the dimension of the $ColA$. (i.e. it's the number of pivot columns in A.)

Example

Consider the matrix:

$$A = \begin{pmatrix} -3 & 7 & 11 \\ 1 & 5 & 0 \\ -4 & -6 & 7 \end{pmatrix} \text{ row equivalent to } \begin{pmatrix} 1 & 0 & -5/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}.$$

What is the rank$A$? TWO! (The dimension of $ColA$ is two. There are two pivot columns.)

Note: **The Rank Theorem**

If a matrix A has n columns, then rank$A$ + dim $NulA$ = n.

* rank$A$ is same as the number of pivot columns.
* dim$NulA$ is the same as the number of free variables, or the number of non-pivot columns.
* So, the theorem....

Basis Theorem

Let H be a p-dimensional subspace of $\Re^n$. Any linearly independent set of exactly p elements in H is automatically a basis for H. Said differently, any set of p elements of H that spans H is automatically a basis for H.

Example (#14)

Find a basis for the subspace spanned by the given vectors. What is the dimension of the subspace.

$$A = \begin{pmatrix} 1 \\ -1 \\ -2 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ -1 \\ 6 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -6 \\ 8 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ -7 \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ -8 \\ 9 \\ -5 \end{pmatrix}.$$

Invertible Matrix Theorem (continued)

Let A be an n x n matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix.

m. The columns of A form a basis of $\Re^n$.

n. $Col A = \Re^n$.

o. dim $Col A = n$.

p. rank$A = n$.

q. $Nul A = \{\vec{0}\}$.

r. dim$Nul A = 0$.