TD01 - Vector bundles

Exercise 1. Let M be a smooth manifold. What are the global sections of $M \times \mathbb{R}^k \to M$, $TM \to M$ and $\bigwedge^k T^*M \to M$?

Exercise 2 (Frames)**.** Let $E \to M$ be a smooth vector bundle of rank k. A *local frame* for E over the open subset $U \subset M$ is a family $(s_1, \ldots, s_k)$ of smooth sections of $E_{|U} \to U$ such that, for any $x \in U$, $(s_1(x), \ldots, s_k(x))$ is a basis of the fiber E_x.

1. Check that it is equivalent to give a local frame for E over U or a local trivialization $E_{|U} \simeq U \times \mathbb{R}^k$.
2. Give a necessary and sufficient condition on global sections of $E \to M$ for this bundle to be trivial.
3. Is $T\mathbb{T}^n \to \mathbb{T}^n$ trivial? Is $T\mathbb{S}^2 \to \mathbb{S}^2$ trivial?
4. Does any smooth vector bundle admit a non-zero smooth section? A non-vanishing smooth section?

Exercise 3 (Pullback)**.**

1. Is the pullback of a trivial vector bundle trivial?
2. Let $\pi : \mathbb{S}^n \to \mathbb{RP}^n$ be the canonical projection, compare $T\mathbb{S}^n \to \mathbb{S}^n$ and $\pi^*(T\mathbb{RP}^n) \to \mathbb{S}^n$.