TD06 - More connections

Exercise 1 (Hessian and torsion)**.** Let M be a manifold and let $f : M \to \mathbb{R}$ be smooth.

1. In a chart (U, φ), we can compute the second diffenrential of $f_\varphi := f \circ \varphi^{-1}$. How do chart transitions act on $D^2 f_\varphi$? Can we define an intrinsic notion of second differential of f that would read as $D^2 f_\varphi$ in the chart (U, φ), for any such chart?

2. Let us now assume that TM is endowed with a connection ∇. Let $\nabla^2 f$ be the section of $T^*M \otimes T^*M$ defined by:

$$\forall x \in M,\ \forall u, v \in T_xM, \quad \nabla_x^2 f(u,v) := (\nabla_u(df)) \cdot v.$$

   Let X and Y be two vectors fields on M, prove that $\nabla^2 f(X,Y) = X \cdot (Y \cdot f) - (\nabla_X Y) \cdot f$.

3. Give an necessary and sufficient condition on ∇ ensuring that, for all $f \in \mathcal{C}^\infty(M)$, for all $x \in M$, $\nabla_x^2 f$ is symmetric.

Exercise 2 (Derivatives at a vanishing point)**.** Let $E \to M$ be a vector bundle and let ∇ and $\widetilde{\nabla}$ be two connections on E. Let $s \in \Gamma(E)$ and $x \in M$ be such that $s(x) = 0$. Compare $\nabla_x s$ and $\widetilde{\nabla}_x s$.

Exercise 3. Let $E \to M$ be a vector bundle eqquiped with a connection ∇. Let $x_0 \in M$ and $y \in E_{x_0}$, is there a local (resp. global) section $s \in \Gamma(E)$ such that $s(x_0) = y$ and $\nabla_{x_0} s = 0$?

Exercise 4. Let s be a smooth section of the vector bundle $E \to M$ and let $x_0 \in M$. Is there a connection ∇ on E such that ∇s vanishes on some neighborhood of x_0 in M?