TD07 - Christoffel symbols and geodesics

Exercise 1 (Christoffel symbols)**.**

1. Let (M, g) be a Riemannian manifold. We denote by $(x_1, \ldots, x_n)$ local coordinates on a open subet U of M and by $G = (g_{ij})$ the matrix of g in these coordinates. Let (g^{kl}) denote the coefficients of G^{-1}. Check that the Christoffel symbols $(\Gamma_{ij}^k)_{1 \leqslant i,j,k \leqslant n}$ of the Levi–Civita connection ∇ of (M, g) are symmetric in (i, j). Prove that for any $i, j, k \in \{1, \ldots, n\}$ we have:

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left(\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right).$$

2. Recall that the half-plane model of the hyperbolic plane is $\mathbb{H}^2 := \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ endowed with the metric $g_{(x,y)} := \frac{1}{y^2}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y)$. Compute the covariant derivatives of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ for the Levi–Civita connection.

3. Recall that the Poincaré disc is the unit open disc $\mathbb{D}^2 \subset \mathbb{R}^2$ endowed with the metric $g_{(x,y)} := \frac{4}{(1-x^2-y^2)^2}(\mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y)$. Compute the covariant derivatives for the Levi–Civita connection of the vector fields $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ associated with the polar coordinates on $\mathbb{D}^2 \setminus \{0\}$.

Exercise 2 (Image of a geodesic)**.** Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be two Riemannian manifolds and let $f : M \to \widetilde{M}$ be a smooth map.

1. If f is an isometric diffeomorphism, is the image of a geodesic of M a geodesic of $\widetilde{M}$?

2. Same question if f is a conformal diffeomorphism.

3. Same question if f is an isometric embedding.