TD08 - Geodesics

Definition. Let (M, g) be a Riemannian manifold. We say that two maximal geodesics are *parallel* if they are either disjoint or equal up to reparametrization.

Exercise 1 (Model spaces)**.**

1. Let us consider $\mathbb{R}^n$ with is canonical Euclidean metric.
   (a) What are the geodesics?
   (b) Compute the exponential map at any point $p \in \mathbb{R}^n$.
   (c) What is its injectivity radius?
   (d) Are there periodic geodesics?
   (e) Are all geodesics periodic?
   (f) Is the image of a geodesic a submanifold of the ambient space?
   (g) Let γ be a geodesic and let $p \in \mathbb{R}^n \setminus \mathrm{Im}(\gamma)$. How many geodesics passing through p and parallel to γ are there?
2. Let $\alpha_1, \dots, \alpha_n > 0$, same questions for $\mathbb{T}^n_\alpha = \mathbb{R}^n/(\alpha_1\mathbb{Z} \oplus \cdots \oplus \alpha_n\mathbb{Z})$ with the metric induced by the Euclidean one on $\mathbb{R}^n$.
3. Same questions on $\mathbb{S}^n$ with the metric induced by the Euclidean one on $\mathbb{R}^{n+1}$.
4. Same questions on the Poincaré disc $\mathbb{D}$ with the metric $g_{\mathbb{D}} := \frac{4}{(1-x^2-y^2)^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$
5. Same questions on the upper half-plane $\mathbb{H}$ with the metric $g_{\mathbb{H}} := \frac{1}{y^2}(\mathrm{d}x^2 + \mathrm{d}y^2)$.