# TD09 - Normal coordinates and tensor fields

**Exercise 1** (Normal coordinates)**.** Let $(M, g)$ ba a Riemannian manifold and let $p \in M$.

1. Can we find local coordinates $(x_1, \dots, x_n)$ centered at $p$ such that the matrix $(g_{ij}(x))$ of $g$ in these coordinates satisfies $(g_{ij}(x)) = I_n + O(\|x\|)$, where $I_n$ is the identity matrix of size $n$?
2. Show that in the normal coordinates centered at $p$ we have: $(g_{ij}(x)) = I_n + O(\|x\|^2)$, and $\Gamma_{ij}^k(x) = O(\|x\|)$ for any $i, j$ and $k$.
3. Can we do better? $(I_n + O(\|x\|^3)$, constant equal to $I_n$, $\dots)$

**Exercise 2.** Let $E$ and $E'$ be two smooth vector bundles over the same base space $M$. Let $F : \Gamma(E) \to \Gamma(E')$ be a $\mathcal{C}^\infty(M)$-linear map.

1. Prove that there exists a unique $f : E \to E'$ which is a bundle map over $M$ (that is, for any $x \in M$, $f_{|E_x} \in \mathrm{End}(E_x, E'_x)$) and satisfies:
$$\forall s \in \Gamma(E), \forall x \in M, \qquad F(s)(x) = f(s(x)). \tag{1}$$
2. Check that $f$ can be seen as an element of $\Gamma(E^* \otimes E')$.
3. Conversely, check that a section $f \in \Gamma(E^* \otimes E')$ defines a unique $\mathcal{C}^\infty(M)$-linear map $F : \Gamma(E) \to \Gamma(E')$ satisfying (1).

**Remark.** In other terms we have defined a canonical isomorphism of $\mathcal{C}^\infty(M)$-modules between $\mathrm{Hom}(\Gamma(E), \Gamma(E'))$ and $\Gamma(E^* \otimes_{\mathbb{R}} E')$.

Similarly, we can show that $s \otimes s' \mapsto (x \mapsto s(x) \otimes s'(x))$ defines a canonical isomorphism of $\mathcal{C}^\infty(M)$-modules from $\Gamma(E) \otimes_{\mathcal{C}^\infty(M)} \Gamma(E')$ to $\Gamma(E \otimes_{\mathbb{R}} E')$. For example, $\Gamma(\bigwedge^k T^*M \otimes_{\mathbb{R}} E)$ is isomorphic to $\Omega^k(M, E) := \Omega^k(M) \otimes_{\mathcal{C}^\infty(M)} \Gamma(E)$, the space of $k$-forms with values in $E$.