

TD10 - Curvature

Exercise 1. Let ∇ be a connection on a smooth vector bundle $E \rightarrow M$. We can see ∇ as a map from $\Omega^0(M, E)$ to $\Omega^1(M, E)$. We extend $\nabla : \Omega^k(M, E) \rightarrow \Omega^{k+1}(M, E)$ by forcing the Leibniz rule:

$$\forall \alpha \in \Omega^k(M), \forall s \in \Gamma(E), \forall x \in M \quad \nabla_x(\alpha \otimes s) = d_x \alpha \otimes s(x) + (-1)^k \alpha_x \wedge \nabla_x s$$

and \mathbb{R} -linearity.

1. Prove that $\nabla \circ \nabla$ is $\mathcal{C}^\infty(M)$ -linear from $\Omega^0(M, E)$ to $\Omega^2(M, E)$.
2. Using the previous exercise, show that $\nabla \circ \nabla$ defines a section $R \in \Omega^2(M, \text{End}(E))$. We call R the *curvature* of (E, ∇) .
3. Let (x_1, \dots, x_n) be local coordinates on M and (e_1, \dots, e_r) be a local frame for E defined on the same open set U . We denote by (Γ_{ij}^k) the Christoffel symbols of ∇ in these coordinates. We also define (R_{ijk}^l) by:

$$\forall i, j, k \in \{1, \dots, n\}, \quad R \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) e_k = \sum_{l=1}^n R_{ijk}^l e_l.$$

Check that for any $i, j \in \{1, \dots, n\}$ and any $k, l \in \{1, \dots, r\}$ we have:

$$R_{ijk}^l = \frac{\partial \Gamma_{jk}^l}{\partial x_i} - \frac{\partial \Gamma_{ik}^l}{\partial x_j} + \sum_{m=1}^r \Gamma_{jk}^m \Gamma_{im}^l - \sum_{m=1}^r \Gamma_{ik}^m \Gamma_{jm}^l. \quad (1)$$

4. Check that for any $X, Y \in \Gamma(TM)$ and any $s \in \Gamma(E)$ we have:

$$\nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X, Y]} s = R(X, Y)s.$$

Remark. In particular, if ∇ is the Levi-Civita connection of (M, g) then R is its Riemann curvature. Equation (1) is of course valid in this case.

Exercise 2. Let (M, g) be a Riemannian manifold of dimension n .

1. If $n = 1$, what is its curvature?
2. If $n = 2$, how many degrees of freedom are there in the Riemann tensor? Give the expression in local coordinates of the Riemann, Ricci and scalar curvature of M .
3. How many degrees of freedom are there in the Riemann tensor for $n = 3$ and $n = 4$.

Exercise 3. What are the Riemann, Ricci and scalar curvatures of \mathbb{R}^n and \mathbb{T}^n ?