TD12 - More more curvature

Exercise 1. Let us consider normal coordinates $(x^1, \ldots, x^n)$ around some point p in a Riemannian manifold (M, g). Let us denote as usual (g_{ij}) the matrix of g in these coordinates, (Γ_{ij}^k) the Christoffel symbols of the Levi–Civita connection and (R_{ijkl}) the components of the Riemann tensor (as a $\binom{4}{0}$-tensor). We admit that the following holds in these coordinates:

$$\forall i, j \in \{1, \ldots, n\}, \qquad g_{ij}(x) = \delta_{ij} - \frac{1}{3} \sum_{1 \leqslant k,l \leqslant n} R_{iklj}(0) x^k x^l + O(\|x\|^3).$$

1. Give a two terms expansion of the Riemannian volume $\mathrm{d}V$ around 0 in these coordinates.
2. Give a two terms expansion of the volume of the geodesic ball of center p and radius ρ as $\rho \to 0$.

Exercise 2. Let (M, g) be a Riemannian manifold and let $f : M \to \mathbb{R}$ be a smooth map. We assume that f vanishes transversally, that $d_x f \neq 0$ whenever $f(x) = 0$. Let us denote by Z the hypersurface $f^{-1}(0)$ and by II its second fundamental form.

1. Show that for any $x \in Z$, $d_x f \circ \mathrm{II}_x = -\nabla_x^2 f_{|T_x Z}$.
2. Express II only in terms of f.

Exercise 3. Recall that if (M, g) is a Riemannian surface, then its Gauss curvature at p is $\kappa(p) = K(T_p M)$, the sectional curvature of $T_p M$.

1. Is there a metric g on $\mathbb{S}^2$ such that κ takes a negative value at some point?
2. Is there a metric g on the torus $\mathbb{T}^2$ such that κ does not vanish?
3. Is there a 2-dimensional submanifold (without boundary) M of $\mathbb{R}^3$ such that κ vanishes everywhere?
4. Same question assuming that M is not a plane.
5. Same question assuming that M is compact without boundary. (*Hint:* Use Sard's theorem: the set of critical values of a smooth map between smooth manifolds has measure 0.)
6. Is there a compact surface without boundary M in $\mathbb{R}^3$ such that κ is negative everywhere?