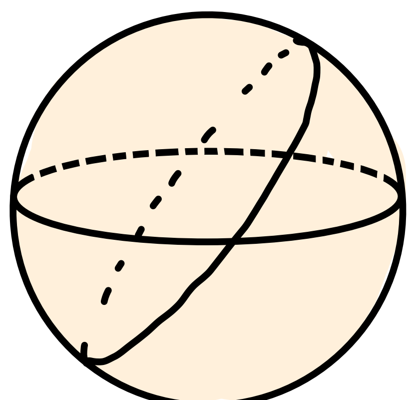
COURBURE

(M,g) variété riemannienne

(M,g) n'est pas forcément localement isométrique à l'espace euclidien

ex



(S^2, g_{ronde})

$\exists$ géod distinctes γ, ξ

tq $t \longmapsto d(\gamma(t), \xi(t))$ constante

∇ Levi-Civita de (M,g)

$$\cancel{\nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{[X,Y]}}$$

$$\nabla_{\partial_{x^i}} \underbrace{\nabla_{\partial_{x^j}} \partial_{x^h}}_{\sum_k \Gamma^k_{jh} \partial_{x^k}} - \nabla_{\partial_{x^j}} \underbrace{\nabla_{\partial_{x^i}} \partial_{x^h}}_{\sum_k \Gamma^k_{ih} \partial_{x^k}} = \sum_k \left(\partial_{x^i} \Gamma^k_{jh} - \partial_{x^j} \Gamma^k_{ih} + \sum_\ell \left(\Gamma^\ell_{ik} \Gamma^k_{jh} - \Gamma^\ell_{jk} \Gamma^k_{ih} \right) \right) \partial_{x^k}$$

$$\nabla_{[\partial_{x^i}, \partial_{x^j}]} = 0$$

TENSEUR DE COURBURE de Riemann

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

$$= \text{"}\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \right) Z \text{"}$$

Rmq R est un $\binom{3}{1}$-champs tensoriel

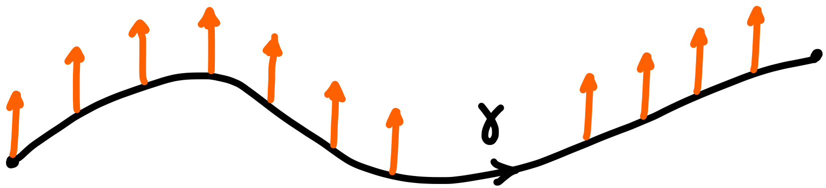
$\forall$ X, Y, Z champs de vecteurs sur M, $f \in C^\infty(M)$

$$R(fX,Y)Z = f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{f[X,Y]-(Yf)X} Z$$

$$= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - \cancel{(Yf)\nabla_X} Z - f\nabla_{[X,Y]} Z + \cancel{(Yf)\nabla_X} Z$$

$$= f\, R(X,Y)Z$$

$$\left(\begin{array}{l} \text{et de façon analogue} \\ \quad R(X, fY)Z = fR(X,Y)Z = R(X,Y)fZ \end{array} \right)$$



$\gamma \ [0,T] \to M$

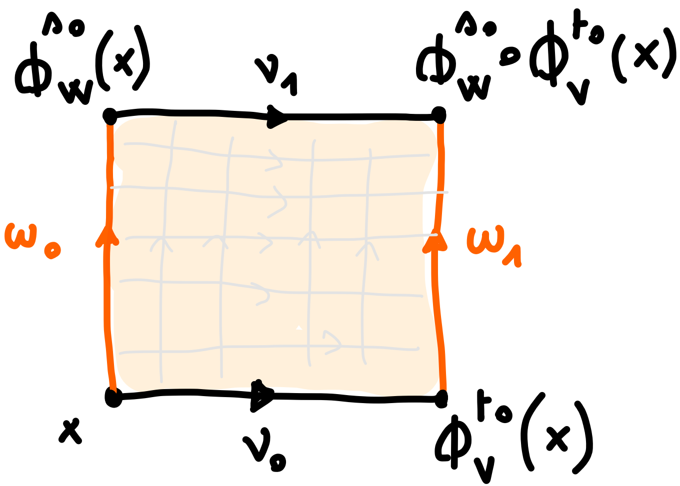
TRANSPORT PARALLEL le long de γ

$$P_s^t \quad T_{\gamma(s)}M \xrightarrow[\cong]{\text{lin}} T_{\gamma(t)}M, \qquad s,t \in [0,T]$$

$$V(s) \mapsto V(t)$$

où V champ de vecteurs le long de γ tq $\underbrace{\nabla_t V \equiv 0}_{\text{EDO d'ordre 1}}$

V, W champs de vecteurs sur M tq $[V,W] \equiv 0$



$$\Rightarrow \underbrace{\phi_V^t}_{\text{flot de } V} \circ \underbrace{\phi_W^s}_{\text{flot de } W} = \phi_W^s \circ \phi_V^t$$

$$w_i(s) = \phi_W^s(\,..\,)$$

$$v_i(t) = \phi_V^t(\,.\,)$$

$P(v_i)_0^{t_0}$ = transport parallèle le long de v_i

$P(w_i)_0^{s_0}$ = transport parallèle le long de w_i

Prop Si $R \equiv 0$, alors $P(v_1)_0^{t_0} \circ P(w_0)_0^{s_0} = P(w_1)_0^{s_0} \circ P(v_0)_0^{t_0}$

Preuve

$$\Sigma = \left\{ \phi_v^t \circ \phi_w^s(x) \mid t \in [0, t_0], s \in [0, s_0] \right\}$$

$0 \neq X(0,0) \in T_x M$ arbitraire

on l'étend à un champs des vecteurs le long de Σ :

$\nabla_v X(t,0) \equiv 0$ i.e. $X(t,0) = P(v_0)_0^t X(0,0)$

$\nabla_w X(t,s) \equiv 0$ $X(t,s) = P(\tilde{\omega}_t)_0^s X(t,0)$

où $\tilde{\omega}_t(s) = \phi_w^s(\phi_v^t(x))$

Il suffit de prouver $\boxed{\nabla_v X \equiv 0}$

On sait $\nabla_v X(t,0) = 0 \quad \forall t \in [0, t_0]$

$$\underbrace{\nabla_w \nabla_v X =}_{R \equiv 0} \nabla_v \underbrace{\nabla_w X}_{0} + \nabla_{\underbrace{[w,v]}_{0}} X = 0$$

$\Rightarrow \quad s \mapsto \nabla_v X(t,s)$ est parallele,

et comme $\nabla_v X(t,0) = 0$ on conclut $\nabla_v X \equiv 0$

$\square$

Thm

$R \equiv 0$ (on dit que (M,g) est plate)

ssi

(M,g) est localement isométrique à $\underbrace{(\mathbb{R}^n, g_{eucl})}_{\text{espace euclidien}}$

i.e. autour de chaque point de M

$\exists$ coordonnées locales $x^1, \dots, x^n$

t.q. $g = dx^1 \otimes dx^1 + \dots + dx^n \otimes dx^n$

Preuve

- Si (M,g) est localement euclidienne, alors $R \equiv 0$
- Supposons $R \equiv 0$ On fixe $q \in M$

$e_1, .., e_m \in T_qM$ base orthonormale de T_qM

Prop précedente $\Rightarrow$ on peut étendre $e_1, ..., e_m$ à des champs de vecteurs $E_1, .., E_m$ tq

$$\nabla E_j \equiv 0 \quad \forall\ j = 1, .., m$$

- $E_1, .., E_m$ lin indép près de q
- $[E_i, E_j] = \nabla_{E_i} E_j - \nabla_{E_j} E_i = 0$

* $\Rightarrow$ $\exists$ coordonnées locales $x^1, .., x^m$ tq. $E_i = \partial_{x^i}$ $\forall i = 1, .., m$ (près de q)

Justification de *

$E_1, \dots, E_n$ lin indep près de q, $[E_i, E_j] \equiv 0$

$$U \xrightarrow{\psi} M \qquad \psi(x^1, \dots, x^n) = \phi_{E_1}^{x^1} \circ \dots \circ \phi_{E_n}^{x^n}(q)$$

$\cap$

$\mathbb{R}^n$

petit voisinage de l'origine

$$d\psi(0) \frac{\partial}{\partial x^i}\Big|_0 = \frac{\partial}{\partial x^i} \phi_{E_i}^{x^i} \circ \underbrace{\dots}_{id} (q)\Big|_0$$

$$= E_i(q)$$

ψ diffeo loc.

qui définit les coordonnées locales $x^1, \dots, x^n$

$$d\big(g(E_i, E_j)\big) = g(\nabla E_i, E_j) + g(E_i, \nabla E_j) = 0$$

$$\Rightarrow \quad g(E_i, E_j) \equiv \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$\Rightarrow \quad g = dx^1 \otimes dx^1 + \dots + dx^n \otimes dx^n$$

$\square$

Symétries de R

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

- $R(X,Y) = -R(Y,X)$
- (1-ère identité de Bianchi)

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

Preuve

$$\begin{aligned} \text{lhs} &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &\quad + \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X \\ &\quad + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y \\ &= \nabla_X [Y,Z] + \nabla_Y [Z,X] + \nabla_Z [X,Y] - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y \end{aligned}$$

$$= [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \qquad \square$$

identité de Jacobi

- $g(R(X,Y)W,Z) = g(R(W,Z)X,Y)$

Preuve

$$\frac{1}{2}[X,Y]\,\|Z\|_g^2 = g(\nabla_{[X,Y]}Z, Z)$$

$$\|$$

$$\frac{1}{2}\left[X(Y\|Z\|_g^2) - Y(X\|Z\|_g^2)\right] = X\,g(\nabla_Y Z, Z) - Y\,g(\nabla_X Z, Z)$$

$$= g(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z, Z) \Rightarrow g(R(X,Y)Z,Z) = 0$$

$$\Rightarrow \quad R(X,Y)Z \perp Z$$

$$g(R(X,Y)W,Z) + g(R(X,Y)Z,W)$$

$$= \underbrace{g(R(X,Y)(W+Z), W+Z)}_{0} - \underbrace{g(R(X,Y)W,W)}_{0} - \underbrace{g(R(X,Y)Z,Z)}_{0}$$

$$= 0$$

$$\Rightarrow \quad g(R(X,Y)W, Z) = g(R(X,Y)Z, W)$$

$$
\underbrace{\begin{array}{l}
g(R(X,Y)W,Z) + g(R(Y,W)X,Z) \\
g(R(Y,W)Z,X) + g(R(W,Z)Y,X) \\
g(R(W,Z)X,Y) + g(R(Z,X)W,Y) \\
g(R(Z,X)Y,W) + g(R(X,Y)Z,W)
\end{array}}_{0}
\underbrace{\begin{array}{l}
+\, g(R(W,X)Y,Z) \\
+\, g(R(Z,Y)W,X) \\
+\, g(R(X,W)Z,Y) \\
+\, g(R(Y,Z)X,W)
\end{array}}_{\begin{array}{c} 2g(R(W,X)Y,Z) \\ + \\ 2g(\underbrace{R(Z,Y)}_{-R(Y,Z)}W,X) \end{array}}
\begin{array}{l}
= 0 \\
= 0 \\
= 0 \\
= 0
\end{array}
$$

$\square$

• (2ème identité de Bianchi)

$$\boxed{(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0}$$

Preuve $x^1, \dots, x^n$ coordonnées normales géodésiques centrées en $q \in M$

Il suffit de considérer $X = \partial_{x^i}$ $Y = \partial_{x^j}$ $Z = \partial_{x^k}$

$$\left(\Rightarrow [X,Y] = [Y,Z] = [X,Z] = 0, \ \nabla_X Y \big|_q = \nabla_X Z \big|_q = \nabla_Y Z \big|_q = 0 \right)$$

$$(\nabla_X R)(Y,Z)W\big|_q = \nabla_X\big(R(Y,Z)W\big) - R(\underbrace{\nabla_X Y}_{0},Z)W - R(Y,\underbrace{\nabla_X Z}_{0})W$$

$$- R(Y,Z)\underbrace{\nabla_X W}_{0}\ \Big|_q$$

$$= \nabla_X\nabla_Y\nabla_Z W - \nabla_X\nabla_Z\nabla_Y W - \nabla_X\nabla_{\underbrace{[Y,Z]}_{0}} W\ \Big|_q$$

$$= \nabla_X\nabla_Y\nabla_Z W - \nabla_X\nabla_Z\nabla_Y W\ \Big|_q$$

$$(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W\ \Big|_q$$

$$= R(X,Y)\underbrace{\nabla_Z W}_{0} + R(Y,Z)\underbrace{\nabla_X W}_{0} + R(Z,X)\underbrace{\nabla_Y W}_{0}\Big|_q = 0$$

$\square$